Unit 3

Proof and Congruence

Building Arguments That Must Be Accepted

9 lessons • Lessons 3.5a, 3.5b, and 3.5c each run one full day • CYU replaces homework throughout

Unit Navigation Instrument

Big Idea: A proof is not a list of facts. It is a chain of conditional statements — each consequent becoming the antecedent of the next — that leads unavoidably from what is given to what is claimed.

Subtopic	Basic	Intermediate	Advanced
Constructions and What They Prove *→ Proof begins with what we can construct*	Can reproduce the steps of a known construction (equilateral triangle, segment copy) with compass and straightedge.	Can justify each step of a construction by citing a postulate, common notion, or previously proven proposition.	Can explain why a construction proves its result — and why that is stronger than a drawing or a measurement.
Postulates vs. Propositions *→ What we accept vs. what we must earn*	Can define both terms and give an example of each.	Can explain the distinction: a postulate is accepted without proof; a proposition is claimed and then proven from what came before.	Can explain why the order of propositions matters — why Proposition 2 must follow Proposition 1 — and connect this to how a deductive system is built.
Triangle Congruence Shortcuts *→ The geometric facts we draw from*	Can identify which shortcut (SSS, SAS, ASA, AAS) applies, or identify what additional information is needed.	Can explain why SSA and AAA are not valid, and construct a counterexample for each.	Can use shortcuts alongside diagram information (vertical angles, parallel lines, reflexive sides) to identify congruent parts in a complex figure.
Reading a Diagram *→ Given vs. earned from the diagram*	Can list information directly given (from labels and marks) and distinguish it from what must be inferred.	Can identify which facts are always available from any diagram (vertical angles, linear pairs, reflexive sides) and which require an explicit given.	Can look at an unmarked diagram and determine exactly what additional given information would be needed to prove a specific conclusion.

Proof Structure *→ How each step earns the next*	Can explain what must be justified in a proof and what does not require justification (givens, postulates, vertical angles, linear pairs, reflexive property).	Can arrange proof steps in correct logical order, explaining how each consequent becomes the antecedent of the next.	Can identify the missing intermediate step in a proof — the claim that must be proven before the final conclusion is reachable — without being told what it is.
Writing Complete Proofs *→ A complete, undeniable argument*	Can complete a proof by filling in missing justifications when the logical structure is provided.	Can write a complete flow chart or paragraph proof given a diagram and the statement to prove, with no additional scaffolding.	Can write a proof where a required intermediate step is not identified — recognizing independently that a sub-claim must first be established — and produce a complete, well-reasoned argument.

The downloadable navigation instrument includes problem numbers aligned to CYU sets for each lesson. Rows 1 and 2 are active from Lesson 3.1. Row 3 activates in Lesson 3.2. Row 4 activates in Lesson 3.3. Rows 5 and 6 activate in Lesson 3.5a and grow through 3.5c. Earlier rows continue to appear as spiral problems throughout.

CPCTC standing course policy: Before writing CPCTC as a justification in any proof, board work, quiz, or assessment, students must first write its full meaning: “Corresponding Parts of Congruent Triangles are Congruent: if two triangles are congruent, then all six corresponding parts (three sides and three angles) are congruent.” The abbreviation is only permitted after the definition has been written. This policy appears on the navigation instrument and in every relevant lesson.

What This Unit Is About

In Unit 2 we built the logical infrastructure: inductive reasoning and its limits, deductive reasoning and its power, conditional statements, the law of syllogism, and the axiomatic system that gives deduction its foundation. We ended by accepting Euclid’s ten axioms as shared starting points. The question that opens this unit is: now what?

Now we use them. This unit is not primarily about geometric facts — though it contains plenty. It is about the act of constructing arguments that cannot reasonably be rejected. Arguments where every step is justified. Arguments where the conclusion must follow from the premises, not just probably does. The geometric content — constructions, congruence shortcuts, transversal relationships — is the raw material from which those arguments are built. The logical architecture is what this unit is really teaching.

That architecture has value far beyond geometry. When students leave this unit understanding how to string together conditional statements — justifying every step, ensuring no gaps, checking that each conclusion earns what follows from it — they have learned how to construct a persuasive essay, how to build a legal argument, how to reason through a design decision, how to think in any domain where making a good case matters. The habit of asking "what am I assuming, and have I earned the right to assume it?" is one of the most valuable intellectual habits a person can develop. Geometry is where we practice it in a setting rigorous enough to make the habit stick.

The unit begins with construction — a word that means something precise here. A construction is not a drawing. A drawing shows what something might look like. A construction, built with compass and straightedge following the postulates exactly, proves the object exists within the axiomatic system. Lesson 3.1 makes that distinction explicit and immediately connects it to proof: the ability to construct something is the ability to prove it is real.

From there the unit builds the toolkit students will draw from in proof writing: triangle congruence shortcuts (Lesson 3.2), diagram reading and CPCTC (Lesson 3.3), and a diagnostic day (Lesson 3.4) that tells both teacher and students exactly where they stand before proof writing begins. The proof sequence itself runs across three lessons: a proof progression method (Lesson 3.5a), group presentations with peer critique (Lesson 3.5b), and scaffolded proof practice with student-chosen entry points (Lesson 3.5c). Review and assessment close the unit.

On definitions and logical argument — a thread that runs through the whole course

It is not a coincidence that Euclid begins his Elements not with axioms but with definitions. He names twenty-three of them before a single postulate is introduced. Definitions are not preliminaries to be dispatched quickly. They are essential to the logical system itself. An argument is only as precise as its terms, and a term used loosely in the middle of a proof is a hidden hole.

This thread begins in Lesson 3.1 with the dragon definition task. Students learn that a precise definition does not guarantee existence — it only tells you what something would be if it existed. A construction must follow to prove it can exist. But the precision of the definition is what makes the construction meaningful.

It continues in Lesson 3.3, where CPCTC is given the same treatment: before writing it as an abbreviation, write what it means. The abbreviation is a tool for communication. The meaning is what justifies its use in an argument. Students who understand the definition can use the abbreviation confidently; students who use the abbreviation without understanding the definition are borrowing authority they have not earned.

It is also why Unit 1's vocabulary work matters more than it might appear. Students who cannot name an angle correctly cannot write a proof that refers to it precisely. Definitions are the foundation of clear argument — not just in geometry, but everywhere.

What Changed From Year 1 and Why

This unit was substantially rewritten after the first year of teaching it. The honest account below is offered because that is what this book is for: not a polished version of perfect teaching, but the real story of what happened and what was learned. Teachers who adopt these materials should expect to make their own revisions. The version in your hands is already the second draft.

The original unit's central weakness was sequencing. Proof writing was introduced after the geometric content, as if proof were a topic that followed from the geometry rather than the thing the whole unit is building toward. Students spent the proof section mimicking steps rather than constructing arguments. Two-column proofs were used throughout and students preferred them — but for the wrong reason. Two-column format allows individually justified statements that do not chain together. It hides the shape of the argument. Students found it easier because it was doing less of the mathematical work.

Two-column proofs are retired entirely in the revised unit. Only flow chart proofs and paragraph proofs are permitted. In a flow chart proof, the law of syllogism is visible: you can see how the output of one box feeds into the next, and how at certain points two boxes converge to justify a single conclusion. In a paragraph proof, the argument reads as connected prose, each claim followed immediately by its justification. Both formats require students to construct a genuine argument, not a list.

The second major change was adding Lesson 3.3 as a dedicated lesson on reading a diagram. Students in Year 1 were missing shared sides and vertical angles because they were not marked in the diagram. They were reading tick marks, not geometry. Lesson 3.3 addresses this directly by presenting deliberately unmarked diagrams and asking students to list everything the diagram guarantees — before any given information is provided.

The third major change was adding Lesson 3.4 as a diagnostic practice day before proof writing begins. Students need to have solid command of the congruence shortcuts and diagram reading before they can build arguments with them. It became clear after Year 1 that many students did not. Moving on to proof structure when the basic tools are not yet understood guarantees struggle — not productive struggle, but the kind that produces frustration and mimicry. Lesson 3.4 gives both teacher and student a clear picture of where they actually are.

CPCTC was being used as a label without understanding. Students wrote it without connecting it to the definition of congruence that established it in Lesson 3.2. The course policy — write the full definition before using the abbreviation, every time, in every context — is the response. It will feel tedious but I think it will be worth the cost.

On the Euclidean Handbook: it was created mid-unit in Year 1 as a patch when students were not tracking which results had been established. It helped, but it arrived as a workaround rather than a designed element. The rewrite addresses the underlying need through better lesson design rather than a separate tracking document. The handbook is retained as an optional resource — it has genuine value as an organizational structure that mirrors how Euclid himself organized the Elements, and any teacher who finds value in having students record propositions in the order they are proven should use it. I may return to it in future years. But the revised lessons are designed so that it is not required.

Lesson Sequence Overview

Lesson 3.1: Constructions as Proof — dragon definition task, Euclid's Proposition 1, nullilateral triangle, drawing vs. construction, postulate vs. proposition

Lesson 3.2: What Makes Triangles the Same — manipulative discovery of SSS/SAS/ASA/AAS/HL, SSA and AAA disproved, CPCTC introduced

Lesson 3.3: Reading a Diagram — free information vs. must be given, CPCTC policy established, Mathmedic poster task

Lesson 3.4: Diagnostic Practice Day — individual work, navigation instrument rows 1–4, self-assessment before proof begins

Lesson 3.5a: Proof Progressions — Ray Allen story, three-stage parallelogram reveal, flow chart format established, two-column proofs retired

Lesson 3.5b: Proof Presentations — group proofs, peer critique, individual follow-up

Lesson 3.5c: Scaffolded Proof Practice — mild/medium/spicy, student-chosen entry points

Lesson 3.6: Review Day — Stella's Stunners, navigation instrument update, tiered problem bank

Lesson 3.7: Unit Assessment — scrambled question order, navigation instrument grading rubric