

Coordinate Proof Day

Unit 5: Quadrilaterals • Day 5 of unit • Groups at boards all day

At a Glance

Pacing: Tier 1 problems to all groups simultaneously (~15 min) • Tier 2 problems given individually as groups are ready (~50 min) • Tier 3 for fast groups (~15 min if reached) • Closing discussion (~5 min)

Nav instrument rows active: Row 5 (Coordinate Methods) — all three levels • All prior rows spiral through the problems

Problem delivery: Give Tier 1 to all groups simultaneously as a warm-up. Move to Tier 2 individually as groups finish. Tier 3 for groups who move quickly — most classes will spend the majority of time in Tier 2.

Descartes story: Worth telling at the start — see the gold box below.

OBJECTIVES

By the end of this lesson students can:

- Use slope to confirm parallel and perpendicular sides.
- Use distance to confirm congruent sides and diagonals.
- Use midpoint to confirm diagonal bisection.
- Classify quadrilaterals using coordinate methods, showing all required checks.
- Explain the sufficient-vs-necessary distinction: showing one property consistent with a classification is not the same as confirming the classification.
- Articulate why coordinate proof and Euclidean proof reach the same conclusions from different foundations.

The Descartes Story — worth telling at the start

The story most often told: René Descartes, lying in bed one morning in 1637, watched a fly crawling across his ceiling and realized he could locate any point using just two numbers — its distance from one wall and from an adjacent wall. Coordinate geometry was born.

It is almost certainly apocryphal. No contemporary source records the fly story; it appears to have entered circulation well after Descartes' death. What is documented: Descartes published his coordinate geometry in 1637 as an appendix to his philosophical work Discourse on the Method. The real motivation was mathematical — he wanted algebra to solve geometric problems, and coordinates were the bridge.

The fly story is worth telling anyway — not as fact, but as a reminder that great mathematical ideas often begin with a deceptively simple question: how do I describe where something is? The coordinate plane is Descartes' answer. Today students use it to do exactly what he intended: prove geometric results with algebraic tools.

 **Narrative Arc** — Open class — say before revealing the first problem

"In Lessons 5.2 through 5.4 you proved quadrilateral properties using triangle congruence, alternate interior angles, and CPCTC. Today you prove the same facts using slope, distance, and midpoint. Same conclusions, different starting points. Watch for when you reach the same result by a completely different route. That is not a coincidence. It is telling you something important."

COORDINATE TOOLS REFERENCE

Post this on the board or wall at the start of class. Do not spend time teaching the formulas — students have them from Unit 1. This is a review reference.

Tool	Formula	What it checks
Slope	$m = (y_2 - y_1) / (x_2 - x_1)$	Equal slopes $\rightarrow$ parallel. Slopes with product $-1 \rightarrow$ perpendicular. Undefined $\perp$ zero $\rightarrow$ perpendicular.
Distance	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Equal lengths $\rightarrow$ congruent sides or diagonals.
Midpoint	$M = ((x_1 + x_2)/2, (y_1 + y_2)/2)$	Same midpoint for both diagonals $\rightarrow$ diagonals bisect each other.

The perpendicular slope check — flag this before Tier 2

Product = -1 : If slope of AB $\times$ slope of BC = -1 , the lines are perpendicular. This is the standard check.

Undefined and zero: A vertical line (undefined slope) is perpendicular to a horizontal line (slope = 0). The product formula fails here — students must recognize this case visually. It comes up in Problems 8 and 10.

TIER 1 — SINGLE TOOL WARM-UP (GIVE TO ALL GROUPS SIMULTANEOUSLY)

Give all three problems to all groups simultaneously. Expect 10–15 minutes. Purpose: confirm fluency with each tool before full classification problems.

Tier 1 — Problem 1 — Parallel Check

Problem:

Points A(–3, 1), B(1, 3), C(0, –2), D(4, 0). Is AB $\parallel$ CD? Show your work.

Answer + coaching notes:

$$\text{Slope AB} = (3-1)/(1-(-3)) = 2/4 = 1/2$$

$$\text{Slope CD} = (0-(-2))/(4-0) = 2/4 = 1/2$$

Equal slopes $\rightarrow$ AB $\parallel$ CD. $\checkmark$

Watch for:

Did they subtract in consistent order ($y_2 - y_1$ over $x_2 - x_1$)?

Did they state the conclusion explicitly — not just that the slopes are equal?

Tier 1 — Problem 2 — Distance Check

Problem:

Points P(-2, 3) and Q(4, -1). Find PQ. Leave in simplest radical form.

Answer + coaching notes:

$$PQ = \sqrt{[(4-(-2))]^2 + (-1-3)^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$$

Watch for:

Did they square the negative correctly? $(-4)^2 = 16$, not -16 .

Did they simplify $\sqrt{52}$ to $2\sqrt{13}$?

Tier 1 — Problem 3 — Midpoint and Bisection Check

Problem:

Quadrilateral ABCD: A(1, 5), B(6, 4), C(5, -1), D(0, 0). Do the diagonals bisect each other?

Answer + coaching notes:

$$\text{Midpoint of AC} = ((1+5)/2, (5+(-1))/2) = (3, 2)$$

$$\text{Midpoint of BD} = ((6+0)/2, (4+0)/2) = (3, 2)$$

Same midpoint $\rightarrow$ diagonals bisect each other. $\checkmark$
 ABCD is a parallelogram (by the diagonal bisection converse from Lesson 5.2).

Watch for:

Did they find BOTH midpoints and compare, not just one?

Did they state what the equal midpoints imply about the quadrilateral?

Ask: is showing equal midpoints sufficient to classify as a parallelogram? (Yes — by the converse proven in 5.2.)

TIER 2 — FULL CLASSIFICATION (GIVE INDIVIDUALLY AS GROUPS ARE READY)

Each problem requires a complete classification argument — not just one check. Give individually. The answer and coaching notes for each are below.

Tier 2 — Problem 4 — Classify the Quadrilateral

Problem:

A(0,0), B(5,0), C(7,4), D(2,4). Classify as specifically as possible. Justify every claim.

Answer + coaching notes:

$$\text{Slope AB} = 0, \text{Slope DC} = 0 \rightarrow AB \parallel DC$$

$$\text{Slope AD} = (4-0)/(2-0) = 2, \text{Slope BC} = (4-0)/(7-5) = 2 \rightarrow AD \parallel BC$$

Two pairs of parallel sides $\rightarrow$ parallelogram.

$$AB = 5, DC = \sqrt{5^2+0} = 5 \checkmark \quad AD = \sqrt{4+16} = 2\sqrt{5} \quad AB \neq AD \rightarrow \text{not a rhombus.}$$

Slope AB = 0, Slope AD = 2: product $\neq -1 \rightarrow$ not perpendicular $\rightarrow$ not a rectangle.

Classification: parallelogram only.

Watch for:

Did they show BOTH pairs of parallel sides? One pair is not enough.

Did they check rectangle (perpendicular sides) and rhombus (equal adjacent sides) before settling?

Did they state the final classification explicitly?

Tier 2 — Problem 5 — Rectangle or Not?

Problem:

$P(-4,0)$, $Q(0,3)$, $R(3,-1)$, $S(-1,-4)$. Is PQRS a rectangle? Justify every step.

Answer + coaching notes:

$Slope\ PQ = 3/4$, $Slope\ RS = 3/4 \rightarrow PQ \parallel RS$
 $Slope\ QR = -4/3$, $Slope\ PS = -4/3 \rightarrow QR \parallel PS \rightarrow$
parallelogram.

$Slope\ PQ \times Slope\ QR = (3/4)(-4/3) = -1 \rightarrow$
perpendicular $\rightarrow$ rectangle. $\checkmark$

Bonus: diagonals $PR = \sqrt{50}$ and $QS = \sqrt{50}$ also confirm rectangle.

Watch for:

Did they establish parallelogram FIRST before checking for rectangle?

Did they use the inverse reciprocal product = -1 for perpendicularity?

Ask: if only the diagonals were equal, would that be sufficient? (Yes — equal diagonals in a parallelogram $\rightarrow$ rectangle.)

Tier 2 — Problem 6 — Rhombus Identification

Problem:

$W(0,2)$, $X(3,0)$, $Y(0,-2)$, $Z(-3,0)$. Prove WXYZ is a rhombus but not a square.

Answer + coaching notes:

$WX = XY = YZ = ZW = \sqrt{13}$ (all equal) $\rightarrow$ rhombus.
 $\checkmark$

$Slope\ WX = -2/3$, $Slope\ XY = 2/3$

$Product = (-2/3)(2/3) = -4/9 \neq -1 \rightarrow$ not
perpendicular $\rightarrow$ not a square.

Watch for:

Did they compute ALL FOUR side lengths?

For 'not a square': did they check adjacent sides for perpendicularity (not diagonals)?

Note: checking unequal diagonals also works to rule out square.

Tier 2 — Problem 7 — Isosceles Trapezoid

Problem:

$A(-3,0)$, $B(3,0)$, $C(2,3)$, $D(-2,3)$. Show ABCD is an isosceles trapezoid.

Answer + coaching notes:

$Slope\ AB = 0$, $Slope\ DC = 0 \rightarrow AB \parallel DC \checkmark$

$Slope\ AD = (3-0)/(-2-(-3)) = 3$, $Slope\ BC =$
 $(3-0)/(2-3) = -3$

AD not parallel to $BC \rightarrow$ exactly one pair of parallel
sides $\rightarrow$ trapezoid. $\checkmark$

$AD = \sqrt{1+9} = \sqrt{10}$, $BC = \sqrt{1+9} = \sqrt{10} \rightarrow$ equal
legs $\rightarrow$ isosceles trapezoid. $\checkmark$

Watch for:

Did they show ONE pair parallel AND the other pair NOT parallel?

Did they compute BOTH leg lengths?

Ask: could they verify instead by checking equal diagonals? (Yes — isosceles trapezoid diagonals are equal.)

Tier 2 — Problem 8 — Kite Identification

Problem:

A(0,4), B(3,1), C(0,-2), D(-3,1). Show ABCD is a kite and verify the perpendicular diagonal property.

Answer + coaching notes:

$AB = AD = 3\sqrt{2}$ (two pairs of consecutive equal sides) $\rightarrow$ kite. $\checkmark$

$CB = CD = 3\sqrt{2}$ (all four sides equal — this is actually a rhombus too!)

Diagonal AC: undefined slope (vertical). Diagonal BD: slope = 0 (horizontal).

Vertical $\perp$ horizontal $\rightarrow$ diagonals perpendicular. $\checkmark$

Watch for:

Did they pair the sides correctly (consecutive, not opposite)?

Note: this kite has all equal sides — it's also a rhombus and a square. A rich discussion opportunity.

The perpendicular check is clean here: undefined $\perp$ zero. The product formula does not apply — flag this.

TIER 3 — ALGEBRAIC COMPLEXITY (FOR GROUPS WHO MOVE QUICKLY)

These involve fractions, negatives, and variable coordinates. Most classes will not reach all of these.

Tier 3 — Problem 9 — Fractional Coordinates

Problem:

P(-1, 1/2), Q(2,2), R(5/2,0), S(-1/2,-3/2). Is PQRS a parallelogram? Classify as specifically as possible.

Answer + coaching notes:

Slope PQ = $(3/2)/3 = 1/2$, Slope SR = $(3/2)/3 = 1/2 \rightarrow PQ \parallel SR$

Slope QR = -4, Slope PS = -4 $\rightarrow QR \parallel PS \rightarrow$ parallelogram.

Slope PQ $\times$ Slope QR = $(1/2)(-4) = -2 \neq -1 \rightarrow$ not a rectangle.

$PQ = (3\sqrt{5})/2$, $QR = (\sqrt{17})/2 \rightarrow$ unequal sides $\rightarrow$ not a rhombus.

Classification: parallelogram only.

Watch for:

Watch for fraction arithmetic errors, especially when adding unlike denominators.

Did they simplify slopes completely before comparing?

Tier 3 — Problem 10 — Negative Coordinates — Square?

Problem:

A(-4,-1), B(-1,-4), C(2,-1), D(-1,2). Prove ABCD is a square.

Answer + coaching notes:

$AB = BC = CD = DA = 3\sqrt{2}$ (all equal) $\rightarrow$ rhombus.

Slope AB = -1, Slope BC = 1. Product = -1 $\rightarrow$ perpendicular $\rightarrow$ rectangle.

Rhombus + rectangle = square. $\checkmark$

Watch for:

Did they show all FOUR equal sides AND perpendicular adjacent sides?

Ask: what is the most efficient set of checks to confirm a square?

Tier 3 — Problem 11 — Variable Coordinates

Problem:

Three vertices of a parallelogram: $A(0,0)$, $B(a,0)$, $C(a+b,c)$ where $a,b,c > 0$. Find vertex D such that $ABCD$ is a parallelogram. Prove using midpoints.

Answer + coaching notes:

Diagonals must bisect each other. Midpoint of $AC = ((a+b)/2, c/2)$.

Set equal to midpoint of $BD: ((a+x)/2, (0+y)/2) = ((a+b)/2, c/2)$.

Solving: $x = b, y = c$. $D = (b, c)$.

Verify: midpoint of $AC = \text{midpoint of } BD = ((a+b)/2, c/2)$. ✓

Watch for:

This requires algebraic rather than numerical reasoning — a significant step up.

Did they set midpoints equal and solve simultaneously?

Ask: does this prove the result for ALL parallelograms of this form, not just one specific case?

Tier 3 — Problem 12 — Challenge — Prove a Property

Problem:

$A(-2,1)$, $B(1,4)$, $C(4,1)$, $D(1,-2)$. (a) Classify. (b) Without computing the intersection point, prove the diagonals bisect each other. (c) Show the diagonals are perpendicular.

Answer + coaching notes:

(a) All four sides $= 3\sqrt{2}$. Adjacent slopes: 1 and -1 , product $= -1$. Square. ✓

(b) Midpoint $AC = (1,1)$. Midpoint $BD = (1,1)$. Equal $\rightarrow$ diagonals bisect. ✓

(c) Slope $AC = 0$ (horizontal). Slope $BD = \text{undefined}$ (vertical). Perpendicular. ✓

Watch for:

Part (b): 'without computing the intersection' means use midpoints, not the actual crossing.

Part (c): horizontal and vertical — perpendicularity is automatic; product formula does not apply.

Ask: did classifying as a square in (a) make (b) and (c) easier?

CLOSING DISCUSSION (≈5 MIN)

Pull the class together. Cover two ideas.

Idea 1 — Sufficient vs. Necessary (say this exactly)

'Showing one pair of parallel sides tells you it might be a trapezoid. Showing both pairs tells you it's a parallelogram. Showing both pairs AND equal adjacent sides tells you it's a rhombus. Each classification requires a specific set of checks — not just any one of them. The question to ask every time: have I shown everything the classification requires, or just something consistent with it?'

Idea 2 — Two Systems, Same Truth (say this exactly)

'In Lessons 5.2 through 5.4, you proved that parallelograms have congruent opposite sides using triangle congruence and CPCTC. Today you proved the same fact using equal distances. In 5.2 you proved diagonals bisect each other using ASA. Today you confirmed it with equal midpoints. Same conclusions. Different starting points.'

'Euclidean geometry begins with postulates and builds through logical deduction. Coordinate geometry begins with the number line and the distance formula. Both systems are internally consistent, and for the geometry we study in this course, they always agree. This is not a coincidence — it is one of the deepest ideas in mathematics. The truth of a geometric fact does not depend on which system you use to prove it.'

STUDENTS MAKE NOTES ON

Students make notes on:

- No separate note sheet this lesson. The CYU is the written record for Row 5.
- The coordinate tools reference (slope, distance, midpoint) if not already in students' notes from Unit 1.
- The sufficient-vs-necessary distinction: write in own words after the closing discussion.
- The two-systems insight: write in own words after the closing discussion.

Narrative Arc — Closing discussion — transition to Lesson 5.6

"You have now proven properties for every major quadrilateral type, twice — once with triangle congruence, once with coordinates. The proofs look different but they arrive at the same place. Next lesson is practice before the assessment. Use your navigation instrument honestly to find where you are on each row. The goal is to leave that lesson knowing clearly what you can do and what you still need to work on."

Connection Points

- Draws from Unit 1: distance formula, midpoint formula, and slope — first introduced in Lessons 1.2 and 1.3
- Draws from Lessons 5.2–5.4: every classification confirmed today corresponds to a theorem proven in those lessons
- The 'two systems, same truth' connection draws from Unit 2's axioms lesson — different starting premises can produce the same logical results
- Feeds into Lesson 5.6 (Advanced Q19): the final problem explicitly asks students to prove a result both ways and reflect on the two systems