

When Size Changes, What Changes With It?

Unit 6: Similarity • Day 6 of unit • The Square–Cube Law and the Biology of Scale

At a Glance

Pacing (90 min block): Launch: why doesn't nature just scale things up? (~5 min) • Task: Bigger Isn't Just Bigger (~30 min) • Whole-class pattern noticing (~15 min) • Biology discussion (~20 min) • Final reflection + CYU (~20 min)

Materials: Boards and markers •

Nav instrument rows active: Row 4 (Scaling Laws: Perimeter, Area, Volume) activates today

Tone note: This is the most interdisciplinary lesson in the course. Lead with curiosity, not coverage. The biology is not being tested — it is the payoff for geometric reasoning.

False pattern warning: Some students will conjecture that area scales by the same factor as side length, especially if they work only with squares early on. This is a valuable wrong conjecture — do not pre-empt it. Let it surface and be disproven.

OBJECTIVES

By the end of this lesson students can:

- Explain why lengths and perimeters scale by k , areas by k^2 , and volumes by k^3 when a figure is scaled by factor k .
- Apply the scaling laws to find unknown dimensions, areas, or volumes given a scale factor.
- Work backwards: find the scale factor or original measurement given a ratio of areas or volumes.
- Explain using the square–cube law why large animals require adaptations that smaller ones do not.
- Articulate why a discrepancy between geometric prediction and empirical observation is not a failure but a scientific opportunity.

Narrative Arc — Launch — before the task

“Imagine a mouse. Now imagine scaling the mouse up to the size of an elephant — same shape, same proportions, just bigger. What goes wrong? Don’t answer yet. Hold the question.”

“In a few minutes you will work out the mathematics. Then we will come back to the mouse.”

THE TASK — BIGGER ISN'T JUST BIGGER (≈30 MIN)

Assign one shape to each group or let groups choose. Include squares, rectangles, triangles, and cubes. Scale factors of 2 or 3 work best. No formulas provided.

Prompt — give verbally

Your shape has been scaled by a factor of 2 (or 3). Without using any formula, figure out what happens to its side lengths, perimeter, area, and — if your shape is 3D — volume. Count, draw, reason. When you have a conjecture, test it on a different scale factor. Be ready to explain why it works, not just what happens.

⚠ Circulation notes

The false pattern: Students working with squares often conjecture that area scales by k . This is wrong but the error is natural — a 2×2 square has 4 unit squares, not 2. Let this surface. Ask: 'Show me why that's true. Draw it out.' The counterexample is in the picture.

Triangle students: harder to count directly — encourage drawing on grid paper or counting triangles within. The scaling pattern holds but is less obvious visually. Worth spending time here.

Cube students: have them count unit cubes explicitly for $k = 2$ (should get 8 unit cubes from 1) and $k = 3$ (should get 27). The pattern k^3 becomes very clear.

Early finishers: ask them to write the general rule as an algebraic expression (side $s \rightarrow ks$; area $s^2 \rightarrow k^2s^2$; volume $s^3 \rightarrow k^3s^3$), then verify it works for $k = 1/2$ (reduction).

WHOLE-CLASS PATTERN NOTICING (≈15 MIN)

Groups share results. Build the pattern together. The goal is for students to articulate why the pattern must be true, not just what it is.

The argument to draw out — use this sequence

Step 1: Length. Each side is multiplied by k . Perimeter is the sum of sides, each multiplied by k , so the perimeter is also multiplied by k . Linear means one dimension.

Step 2: Area. Area is length $\times$ length. Each dimension is multiplied by k . So area is multiplied by $k \times k = k^2$. Two dimensions.

Step 3: Volume. Volume is length $\times$ length $\times$ length. Each dimension multiplied by k . Volume multiplied by k^3 . Three dimensions.

The number of dimensions is the exponent. This is not a coincidence — it is the definition of what dimension means in measurement.

Measurement	Dimensions	Scale factor k	Scales by	Why
Length / Perimeter	1D	k	k^1	Linear — each side multiplied by k .
Area / Surface area	2D	k	k^2	Area = length $\times$ length. Each dimension scales by k , so product scales by k^2 .

Volume	3D	k	k^3	Volume = length × length × length. Three dimensions each scale by k, so product scales by k^3 .
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BIOLOGY DISCUSSION — WHY DOESN'T NATURE JUST SCALE THINGS UP? (≈20 MIN)

Return to the mouse question from the launch. Work through the biology cases using the scaling laws students just established. This should feel like payoff, not new content.

How to frame the discussion

'You now have a tool. Let's use it. Heat production in an animal depends on its volume. Heat loss depends on its surface area. We just established that when you scale by k, volume grows by k^3 and surface area grows by k^2 . What does that mean for a large animal?'

Let students work through the implication before naming it: volume grows faster than surface area, so large animals generate proportionally more heat than they can shed. The geometry alone predicts the problem.

Case study	The geometric problem	What nature did about it
Elephant ears	Heat is produced by volume (k^3) but lost through surface area (k^2). Large animals produce disproportionately more heat than they can lose through skin alone.	Elephant ears are filled with blood vessels and act as radiators, dramatically increasing effective surface area for heat loss.
Insect size limit	Oxygen enters through surface openings but must reach all tissue (volume). As insects grow, the distance oxygen must diffuse increases. Diffusion is efficient only over short distances.	Insects stayed small in modern atmosphere. In the Carboniferous (higher O_2), larger insects could exist because increased O_2 concentration compensated for diffusion distance.
Bone thickness	Bone and muscle strength scale with cross-sectional area (k^2). Body weight scales with volume (k^3). Large animals are proportionally heavier than they are strong.	Large animals evolved disproportionately thick legs and different body plans. A perfectly similar giant human would collapse under its own weight.
Kleiber's Law	If metabolism were governed by heat loss alone, it should scale with surface area: $\propto \text{mass}^{(2/3)}$. Empirical measurement gives $\text{mass}^{(3/4)}$.	The discrepancy points to fractal branching in circulatory and respiratory networks — geometry that allows more effective internal surface area inside a fixed volume.

Kleiber's Law — the discrepancy as opportunity

Geometric prediction: if metabolism is constrained by surface area, it should scale as $\text{mass}^{2/3}$. Empirical result: it scales as $\text{mass}^{3/4}$. This is not a failure of geometry — it is a signal that internal structure matters.

The fractal branching of circulatory networks allows large amounts of effective internal contact area to be packed into a fixed volume. This is a more sophisticated geometry — one that allows organisms to partially escape the square–cube constraint.

Name the principle explicitly: when mathematical prediction and empirical observation differ, the right response is not to discard the mathematics. It is to ask what the mathematics is missing. This is how Kleiber's Law led to fractal biology, and how the anomalous orbit of Uranus led to the discovery of Neptune.

FINAL REFLECTION — WHY DO WE LEARN THIS? (≈10 MIN)

A speech worth giving — adapt in your own words

'Students often ask: when am I ever going to need this? The honest answer is: we don't know, and that's not the right question. The ideas we studied in this unit were developed long before anyone knew they would explain heat loss in elephants, oxygen limits in insects, or how to discover new planets. Mathematical ideas become useful after they exist — and we often cannot predict when or where that will happen.'

'Learning mathematics is not about collecting answers. It is about developing a way of thinking that lets you recognize patterns, make connections, and reason carefully when new situations arise. The square–cube law is not a formula you need to memorize. It is an example of what geometric reasoning can do when you take it seriously.'

CYU 6.5 — ANSWER KEY

Problems 1–4 are Mild (direct application), 5–8 are Medium (reverse or algebraic), 9–12 are Spicy (biological reasoning).

Q	Answer	Teacher note
1	$k = 3$. Perimeter scales by 3 (from 16 to 48 cm). Area scales by 9 (from 16 to 144 cm^2).	Straightforward. Both parts in part b matter — push students to state both.
2	$k = 3$. Volume scales by $k^3 = 27$ (from 8 to 216 cm^3).	Common error: students say volume scales by 3. Insist on k^3 language.
3	Area scales by $k^2 = 25$. Larger area is 25 times the smaller.	Ask: why squared? Connect to area being 2D.
4	Each side scales by 3. Perimeter also scales by 3 (sum of sides, each scaled by k).	The perimeter scaling should feel obvious once students trust the pattern.
5	$V = 40 \times 2^3 = 40 \times 8 = 320 \text{ cm}^3$.	Watch for students who multiply by 2 instead of 2^3 .
6a	Length ratio = $\sqrt[3]{(9/1)} = 3 : 1$.	Connect: $SA \propto k^2$, so $k = \sqrt[3]{(SA \text{ ratio})}$.
6b	Volume ratio = $k^3 = 27 : 1$.	From part a, $k = 3$, so volumes are in ratio 3^3 .

7	Smaller: x^3 . Larger: $(3x)^3 = 27x^3$. Larger is 27 times bigger.	Algebraic case of the same pattern — confirms the formula generally.
8	SA of larger = $2 \times 4^2 = 32 \text{ m}^2$. Volume scale factor = $4^3 = 64$.	SA scales by $k^2 = 16$; Volume scales by $k^3 = 64$.
9a	$k = \sqrt[3]{3}(27/1) = 3 : 1$.	Connect: $V \propto k^3$, so $k = \text{cube root of volume ratio}$.
9b	SA ratio = $k^2 = 9 : 1$.	From $k = 3$.
10	SA scales by $2^2 = 4$. Volume scales by $2^3 = 8$. Volume grows twice as fast as SA. Heat production (volume) outpaces heat loss (surface area) — large animal overheats.	This is the core biological argument. Students should articulate both the math and the consequence.
11	Heat production scales by $k^3 = 27$. Heat loss scales by $k^2 = 9$. Heat production grows $3\times$ faster. Biologically: animal produces far more heat than it can lose, creating overheating risk.	The ratio of production to loss is $k^3/k^2 = k$. Each time size increases by factor k , the heat balance worsens by factor k .
12	Oxygen enters through surface area (k^2), must supply all tissue (volume, k^3). As insect grows, oxygen demand outpaces delivery. Carboniferous period: higher atmospheric O_2 ($\approx 35\%$ vs 21% today) compensated for this constraint, allowing giant insects.	Best answers name both the geometric argument and the historical evidence.

STUDENTS MAKE NOTES ON

Students make notes on:

- Scaling laws table: length/perimeter scales by k , area by k^2 , volume by k^3 — with the dimensional argument for why
- One worked example from the task (their own shape)
- Optional: one biology connection in their own words (why elephants have large ears, or why insects have a size limit) — only if students want to record it. Not required.

Note on biology notes

I recommend not taking notes on the biology content. The geometry is what belongs in students' notes. The biology is the motivation and the payoff — it should be discussed and remembered as a story, not recorded as bullet points.

Narrative Arc — End of lesson — transition to Lesson 6.6

"You have now seen similarity from four directions: as a transformation (dilation), as a condition (equal angles, proportional sides), as a tool (indirect measurement), and as a law (the square–cube law). Lesson 6.6 is the review and practice session before the unit assessment. You will use all four perspectives."

Connection Points

- Draws from Lesson 6.2: the scale factor k established in 6.2 now applies to area and volume, not just length — extending the idea rather than introducing a new one
- Draws from Unit 1: the area formulas for rectangles, triangles, and the volume formula for cubes are prerequisites for the scaling task
- The Carboniferous period oxygen connection draws from real science — if students raise it, confirm it and name it as an example of scientific reasoning
- Feeds into Lesson 6.6: CYU 6.5 problems 9–12 are Advanced-level and may appear in the review
- The fractal biology note (Kleiber's Law) is a genuine mathematical topic — curious students can be directed toward fractal geometry as a future mathematical idea