Lesson 4.1 — Teacher Reference Notes

Triangle Angle Sum, Exterior Angles, and Polygon Sums

Unit 4: Triangle Theorems • Day 1 of 6

At a Glance	
	Pacing: Tasks 1–4 board work (~55 min, given individually as groups are ready) • Gallery walk if time allows • Teacher-guided notes (~25 min, priority over gallery walk) • CYU begins if time allows (~10 min) **Materials:** Boards and markers • Scissors, paper for folding (Task 1 warm-up) • Student note sheet (distribute for notes phase) **Nav instrument rows active:** Row 1 (Triangle and Polygon Angle Relationships) activates today — all three levels **Priority if time runs short:** (1) teacher-guided notes, (2) gallery walk. CYU is homework. **Task pacing:** Tasks are given individually to groups as they are ready — groups do not move through them together. No formal gallery walk until all groups have reached Task 4.

OBJECTIVES

By the end of this lesson students can:

- State the triangle angle sum theorem and the exterior angle theorem as conditional statements.
- Prove the triangle angle sum theorem using a parallel line through the opposite vertex.
- Explain honestly what that proof assumes and why the full chain closes only after Lesson 4.2.
- Derive the exterior angle theorem as a direct consequence of the triangle angle sum theorem.
- Derive the polygon interior angle sum formula using at least one method and connect both methods algebraically.

> **Narrative Arc** — Open class — before revealing any tasks
>
> *“In Unit 2 we accepted 10 axioms. In Unit 3 we proved things using those axioms. Today we ask what the triangle angle sum actually is — and whether we can prove it from what we already have. You probably know the answer. The interesting question is why it’s true.”*

THE CENTRAL EPISTEMOLOGICAL THEME

> **You – the teacher – will want to know this before class.. Name it explicitly during the notes phase.**
>
> Proof approaches will emerge from the board work that are genuinely close to a complete proof, for example:

Altitude from the opposite vertex: Creates two right triangles. Valid, but depends on the altitude construction (Proposition 12 — perpendicular from a point to a line), not yet proven.

Parallel line through the opposite vertex: This is the proof path. Cleaner argument, uses alternate interior angles from Unit 3. But it depends on constructing a parallel through a given point (Proposition 31) — established in Lesson 4.2, not today.

Both give a conditional proof: if the construction is possible, then the angle sum is 180°. That conditional proof is real mathematics — and it is likely exactly what Euclid had in mind when sequencing Proposition 31 before Proposition 32.

Playfair's axiom (Postulate 5) guarantees the parallel line exists — but says nothing about constructing it with compass and straightedge. That is a separate claim, and it is Lesson 4.2's job to establish it.

BOARD TASKS (≈55 MIN, GIVEN INDIVIDUALLY AS GROUPS ARE READY)

Task 1 — Conjecture and Proof Attempt

Say at the start of class. Give no further instructions.

Prompt

I'm curious whether there are any universal truths about the interior angles of a triangle. Experiment with triangles. Form a conjecture. Use paper, scissors, folding, rotating, straightedges, and any propositions or axioms we have already proven. Test on acute, obtuse, right, scalene, and isosceles triangles. If your conjecture holds, find a way to convince someone it must be true for every triangle — not just the ones you drew. If you can't find a proof, write down exactly what you would need in order to prove it.

⚠️ **What to watch for and how to respond**

Groups who know the answer is 180°: Don't confirm. Ask: 'Are you sure this works for every triangle? Could there be one that fails?' Push toward the proof, not the answer.

Cutting a square in half: Valid for right triangles — students can justify why a square is 360° (four right angles). We had a student refines this to 'only isosceles right triangles, and celebrated it explicitly: 'That is a mathematician's correction. You found the boundary of your own argument.'

Folding a triangle to form a parallelogram: Clever, but requires knowing the parallelogram angle sum. Worth noting as a conditional argument.

Drawing an altitude: Creates two right triangles. Legitimate proof path — but depends on the altitude construction (Proposition 12). Note the dependency when checking in with the group.

Drawing a parallel line through the top vertex: The proof path to build toward. If a group finds it, note their board for the gallery walk. If no group finds it after sustained effort, give the hint below.

Hint (give to individual groups only, not the whole class): *'What if you drew a line through the top vertex, parallel to the base? What do you know about the angles it creates?'*

Task 2 — Write the Conditional Proof (give individually once group has the parallel line approach)

Prompt

You've found the argument. Now write it as a formal proof. State clearly what is given and what you are proving. Assume you can construct the parallel line — write that assumption explicitly, don't treat it as free. Cite every reason. Which theorems from Unit 3 are you using? Where exactly does the chain from our axioms break? What would close it?

What to look for in group proofs

Key moves: (1) naming the assumption explicitly, (2) citing the alternate interior angles theorem from Unit 3 by name, (3) recognizing that the linear pair at C closes the argument once the parallel line is in place.

Watch for: Groups who write 'by Playfair's axiom.' Prompt: 'Does Playfair's axiom tell us how to construct the line, or only that it exists?'

Task 3 — Exterior Angle (give individually once group has the conditional proof)

Prompt

Extend one side of your triangle beyond the triangle. A new angle is formed outside. What can you say about this exterior angle? Make a conjecture. Test it on different triangles. Can you prove it using what you just established?

Most groups arrive quickly at: the exterior angle equals the sum of the two non-adjacent interior angles. The proof is a one-step consequence: the exterior angle and the adjacent interior angle form a linear pair (180°), and the three interior angles also sum to 180°, so the exterior angle must equal the sum of the other two. Every group with the conditional proof on their board already has everything they need.

Task 4 — Polygon Angle Sums (give individually once group has the exterior angle result)

Prompt

Triangles have been hiding inside a lot of the geometric arguments we've made. What might the triangle angle sum theorem suggest about the interior angles of other polygons? Experiment with quadrilaterals, pentagons, hexagons. Is there a pattern? Can you write a general formula for an n-sided polygon? Can you explain why it works?

Two methods will likely emerge — plan for both

Method 1 — Triangulate from one vertex: Groups draw all diagonals from a single vertex, find $n - 2$ triangles, arrive at $180(n - 2)$.

Method 2 — Center point: Groups place a point inside and connect to all vertices, find n triangles, subtract the 360° at the center, arrive at $180n - 360$.

If both methods appear: have both groups present during the gallery walk, then confirm algebraically at the board that the two formulas are identical: $180(n - 2) = 180n - 360$. This is one of the strongest moments in the lesson — independent correct reasoning converging on the same truth.

GALLERY WALK (ONLY IF TIME ALLOWS — NOTES ARE THE PRIORITY)

If time allows after all groups have completed Task 4, direct attention to three things:

- A group that cut a square in half: 'What kind of triangle does this prove it for? What did they notice about the limit?'
- A group with a conditional proof: 'Read Step 1 aloud. What are they assuming? What will it take to remove that assumption?'
- Two groups with different polygon formulas: 'Are these actually different? Let's check.'

TEACHER-GUIDED NOTES (PRIORITY OVER GALLERY WALK)

Teacher leads. Students record on their note sheets. This is not group note-making — it is teacher-led with students writing. Before beginning the proofs, pose this question:

Before the notes — honest accounting

'Every proof approach we found today assumed we could construct something we haven't yet proven we can build. What were those constructions?'

Students identify: the altitude (depends on Proposition 12) and the parallel line through a vertex (depends on Proposition 31, established in Lesson 4.2). What we have today is a conditional proof. Lesson 4.2 closes the gap.

Name the historical parallel explicitly: Euclid almost certainly understood the triangle angle sum before he had formally proven the parallel construction. But he could not include it in the Elements until the foundation was in place. Proposition 31 (construct a parallel through *a point) appears before Proposition 32 (triangle angle sum) in the Elements — not because he discovered them in that order, but because the logical chain required it. Your students are living through exactly that experience.*

Triangle Angle Sum Theorem — Conditional Proof (Teacher Reference)

#	Statement	Justification
Given:	Triangle ABC. Angles α at A, β at B, γ at C.	
Assume:	We can construct a line l through C parallel to AB. (Proposition 31 — established in Lesson 4.2.)	*Named explicitly as an assumption.*
1	Construct line l through C parallel to AB.	*By assumption (Proposition 31).*
2	Angle on left side of C (between l and AC) equals α.	*Alternate interior angles: l ∥ AB, transversal AC. (Unit 3.)*
3	Angle on right side of C (between l and BC) equals β.	*Alternate interior angles: l ∥ AB, transversal BC. (Unit 3.)*
4	$\alpha + \gamma + \beta = 180°$ (the three angles at C lie along line l).	*Linear pair theorem — angles on a straight line sum to 180°.*
5	$\alpha + \beta + \gamma = 180°$.	*Therefore, the sum of the interior angles of triangle ABC is 180°.*

Conditional form: If a polygon is a triangle, then the sum of its interior angles is 180°. (Complete once Lesson 4.2 closes the construction gap. ■)

Exterior Angle Theorem — Proof (Teacher Reference)

#	Statement	Justification
Given:	Triangle ABC. Side BC extended to D, forming exterior angle ∠ACD.	
1	∠ACB + ∠ACD = 180°.	*Linear pair theorem.*
2	∠A + ∠B + ∠ACB = 180°.	*Triangle angle sum theorem (just proven).*
3	∠ACB + ∠ACD = ∠A + ∠B + ∠ACB.	*Both equal 180°. (Common Notion 1.)*
4	∠ACD = ∠A + ∠B.	*Subtract ∠ACB from both sides. (Common Notion 3.)*

Conditional form: If a side of a triangle is extended, then the exterior angle equals the sum of the two non-adjacent interior angles. ■

POLYGON INTERIOR ANGLE SUM — BOTH METHODS

Method 1 — Triangulate from one vertex	Method 2 — Center point
Draw all diagonals from one vertex of an n-gon.	Place a point inside the polygon. Connect to all vertices.

Number of triangles formed: n − 2 **Interior angle sum = 180(n − 2)**	Number of triangles formed: n. Subtract the 360° at the center. **Interior angle sum = 180n − 360**
These are the same formula: *180(n − 2) = 180n − 360* ✓	

STUDENTS MAKE NOTES ON

✎ **Students make notes on:**

- Triangle angle sum theorem: conditional form and conditional proof (with assumption named explicitly)
- What the proof assumes and why the full chain closes only after Lesson 4.2
- Exterior angle theorem: conditional form and proof (citing the triangle angle sum theorem)
- Polygon interior angle sum: both methods if both appeared, with the algebraic confirmation that they are identical

Connection Points	• Draws from Unit 2: alternate interior angles theorem from Lesson 2.5 is the key step in the triangle angle sum proof • Draws from Unit 3: the proof structure (conditional, justification for every step) is the same structure used throughout Unit 3 • Draws from Unit 2: the distinction between Playfair's axiom (existence) and Proposition 31 (construction) mirrors the existence vs. constructibility lesson from Lesson 3.1 • Feeds into Lesson 4.2: the conditional proof from today is completed once the parallel line construction is established