Lesson 8.2 — Teacher Reference Notes

All Circles Are Similar

Unit 8: Circles • Day 2 of 7

At a Glance	
	Pacing: Opening 2-beat refresh (~5–7 min) • Task 1: Three Triangles (~12–15 min) • Share-out (~3–4 min) • Task 2: Circles (~18–20 min) • Consolidation (~10 min) • Gallery walk (~5 min) • Notes + CYU (~25 min) **Materials:** Compass and straightedge for each student • Triangle Task 1 handout or projection (Triangles A, B, C in the coordinate plane) • GeoGebra optional for consolidation **Nav instrument rows active:** Row 1 and Row 2

OBJECTIVES

By the end of this lesson students can:

- State both the Unit 6 (corresponding angles and proportional sides) and the transformations (rigid motions and dilation) definitions of similarity, and explain that they are equivalent.
- Use the transformations frame to describe why two equilateral triangles of different sizes are similar, and why an equilateral and an isosceles triangle are not.
- Give the two-step argument that any two circles are similar (shift center, then dilate by r_2/r_1) without a template.
- Explain why this argument works for circles but not for triangles or rectangles, by referencing the number of free parameters.
- Connect circle similarity to the universality of π and preview why arc length and sector area (Lesson 8.3) will be proportional reasoning, not new memorization.
- Describe the rotational and reflective symmetry of a circle, and explain why neither was needed in the similarity proof.

Opening — Two Beats Before Task 1

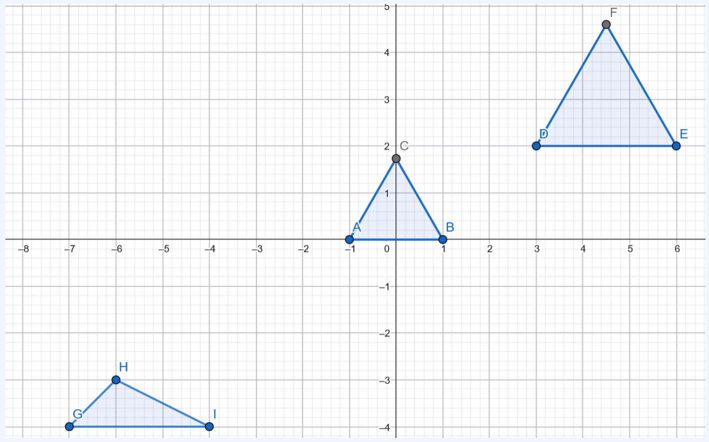
Beat 1 — Refresh Unit 6. Ask: “In Unit 6, when we said two shapes were similar, what did we mean? What did you have to check?” Take 1–2 minutes to collect answers. You are looking for: corresponding angles equal, corresponding sides in a constant ratio (scale factor). Write both conditions on the board as “the Unit 6 definition.” Do not elaborate.

Beat 2 — Hand them the new frame. Say: “That definition works well for shapes with sides and corners. But what are the corresponding sides of two circles? The Unit 6 definition is going to feel awkward today, not wrong, awkward. So, I want to give you a second way of saying ‘similar.’ Two figures are similar if one can be moved onto the other by a sequence of moves that preserve shape — shifting, rotating, reflecting — followed by a dilation. You already know dilation from Unit 6. The new pieces are shifting, rotating, reflecting. Formal names come later in the year; everyday language is fine for now.”

TASKS

📋 Task 1 — Three Triangles (≈12–15 min)

Project or hand out three triangles in the coordinate plane:



The task: *“Using only shifts, rotations, reflections, and dilations, which pairs can be mapped onto each other? For any pair that can, describe the sequence of moves. For any pair that can't, explain why not. What specifically fails?”*

Share-Out — ≈3–4 min between tasks

Pick one or two groups who articulated a clean sequence for A→B and one group who named why A→C fails. Keep it fast. You are not consolidating Task 1 fully, you are surfacing the vocabulary and the “something fails” idea before pivoting to circles. No formal notes yet.

📋 Task 2 — Can You Build Two Circles That Are Not Similar? (≈18–20 min)

Give the task verbally, with no preamble:

“In Unit 6 we discovered that not all quadrilaterals are similar. You just showed that not all triangles are similar either. Your task now: using a compass, try to draw two circles that are not similar. If you think it is impossible, you need to convince me why.”

What you will see: Students arrive at this task already fluent in the transformations frame from Task 1, so the procedural answer (“shift and dilate, it works”) is available much sooner than it would be otherwise. That is a feature, not a bug, it frees their time for the real prize. The linger point is the move from Stage 3 (“it always seems to work”) to Stage 4 (“it must always work because a circle has only one free parameter”).

Mid-task prompt for the whole class (8–10 min in): *“We know not all rectangles are similar. Triangle A and Triangle C couldn't be mapped either. Circles can always be mapped. What is it about circles that rectangles and non-equilateral triangles don't have?”*

The nudge for groups stuck at Stage 2: “What is a circle allowed to be? What parameters describe it?”

⚠ **Do not let groups stop at "shift and dilate, done"**

Because students have the transformations frame from Task 1, the procedural answer is much more available than the conceptual one. A group that has produced "we shifted, we dilated, it worked" has done the easy half. The lesson's whole intellectual payoff sits in the next move: *why must it always work?* If you let groups stop at the procedure, the consolidation will land as a teacher-supplied fact rather than as a student-discovered argument — and the one-parameter idea will not transfer to the rest of the unit. The nudge is some version of: "You have shown it works for two specific circles. What about every possible pair? What is it about circles that makes this inevitable?"

CONSOLIDATION — THE TWO-STEP ARGUMENT AND THE EQUIVALENCE

After the gallery walk, bring the class together. Select two or three groups to share their arguments — prioritize groups that reached definition-based reasoning. Then lead the class through the formal version of the argument, writing it on the board step by step. Before writing, name the equivalence aloud:

"You used the new definition of similarity (shift and dilate) to show all circles are similar. If we check the old definition too: after dilation, every length scales by the same factor r_2/r_1, and every direction from the center rotates consistently. Both definitions agree. They are saying the same thing in different languages."

The Argument — for the Board (Teacher Reference)

Claim: Any two circles are similar.

Let circle C_1 have center A and radius r_1.
Let circle C_2 have center B and radius r_2.

Step 1: *Shift C_1 so its center moves to B. Now C_1 has center B and radius r_1. Call this C_1'.*

Step 2: *Dilate C_1' from center B with scale factor r_2/r_1. Every point on C_1' is at distance r_1 from B; after dilation every point is at distance $r_1 \cdot (r_2/r_1) = r_2$ from B. That is exactly C_2.*

Since C_1 was mapped onto C_2 using a shift and a dilation, C_1 and C_2 are similar. ■

Three questions to ask after the argument:

- *"Did this argument depend on the specific values of r_1, r_2, or the positions of A and B?"*
- *"What would fail if we tried to run this argument for Triangle A and Triangle C from earlier?"*
- *"What is special about circles that means orientation is irrelevant?"*

Question	Answer you are listening for
Generality	No. The argument works for any positive r_1, r_2 and any positions A, B. The scale factor r_2/r_1 is always a well-defined positive number — radii are positive by definition. Worth naming once.
Triangle contrast	Triangles have more than one free parameter. Even after shifting and rotating to align one side, the side ratios do not all match, so no single dilation fixes everything. A triangle's shape is not determined by a single number.
Symmetry	A circle looks identical after any rotation or reflection about its center, because every point is the same distance from the center regardless of direction. Triangles have a preferred orientation — a "top" vertex — so they need rotation. A circle has infinitely many lines of symmetry and infinite rotational symmetry; orientation is simply not a thing two circles can disagree on.

What this closes — and what it opens

The one-parameter idea is the real intellectual takeaway, and it is portable. It explains why all equilateral triangles are similar, why all squares are similar, why all regular n-gons (for fixed n) are similar. Circles are not anomalous, they are the cleanest case of a general pattern: any shape determined by a single parameter has all copies similar to each other.

This opens Lesson 8.3 directly. Arc length and sector area formulas are going to look obvious because all circles are similar, what is true for one is true for all of them, scaled appropriately. Students will not need to memorize the formulas; they will need proportional reasoning, and they already have that. Plant this seed explicitly before they leave: “Tomorrow is going to feel like Unit 6 doing Unit 8's work for us.”

Connection to π and the unit circle — name this aloud

Every circle is similar to the unit circle (radius 1) with scale factor equal to its radius. This is exactly why π appears in formulas for all circles. π is a property of the unit circle, and because all circles are similar to it, π governs them all. Say to students:

“The formula $C = 2\pi r$ is not a separate fact for each circle. It is one fact about the unit circle ($C = 2\pi$ when $r = 1$), scaled up by the similarity ratio r.”

▶ Narrative Arc — End of consolidation — transition to Lesson 8.3

“We proved something universal today. Not ‘these two specific circles are similar,’ but ‘any two circles are similar.’ The proof depended on a single fact about circles: they are determined by one number, their radius. That is what makes them unique among the shapes we have studied.”

“Tomorrow we are going to use this. The formulas for arc length and sector area are going to look almost obvious because all circles are similar, what is true for one is true for all of them, scaled appropriately. You will not need to memorize new formulas. You will need proportional reasoning, and you already have that.”

✎ Students make notes on:

- Two equivalent definitions of similarity: Unit 6 (corresponding angles equal, sides in constant ratio) and transformations (shift, rotate, reflect, dilate). Both describe the same relationship; transformations is the natural frame for circles.
- What Task 1 showed about the three triangles: A and B are similar (shift, rotate, dilate by 3/2); A and C are not similar; what specifically fails for A and C.
- The formal two-step argument that any two circles are similar, in the student's own words.
- Why the argument works for circles but not for triangles or rectangles — the one-parameter idea, stated explicitly.
- Two consequences of all circles being similar: the universality of π, and the forward connection to arc length and sector area.
- The symmetry of a circle (infinite rotational and reflective symmetry) and why this means orientation was irrelevant in the proof — in contrast to Triangle B, where rotation was required.

Connection Points	**Draws from Unit 6:** The Unit 6 definition of similarity (corresponding angles, proportional sides) is the foundation students bring in. The scaling laws from Unit 6 (lengths scale by k, areas by k^2, volumes by k^3) will be the engine of the Extension C derivation of $C = 2\pi r$, $A = \pi r^2$, etc., for any group that reaches it. **Draws from Lesson 8.1:** The definition of a circle as the set of points equidistant from a center is the single fact the whole proof rests on. The one-parameter idea is just that definition, read carefully. **Sets up Lesson 8.3:** Arc length and sector area as proportional reasoning, not new formulas. The seed is the unit-circle connection at the end of consolidation.