Lesson 8.4 — Teacher Reference Notes

Central and Inscribed Angles

Unit 8: Circles • Day 4 of 7

At a Glance	
	Pacing: Launch — broken shard (~10–12 min) • Thinking task — discover inscribed angle relationship (~25–30 min) • Consolidation — proof + corollaries + paper corner (~20–25 min) • Notes (~10 min) • CYU (~10–12 min) (this actually took 2 days) **Materials:** Compasses, rulers, protractors (one each per group) • Printed/cut "broken shard" arcs (200°–220° fragments of a large circle, one per group, no center marked) **Nav instrument row active:** Row 4 (Central and inscribed angles → The spine of the unit) activates today. This row is the engine of nearly every theorem that follows.

OBJECTIVES

By the end of this lesson students can:

- Define central angle, inscribed angle, and intercepted arc, with diagrams.
- State the inscribed angle theorem and sketch its proof (Case 1: center inside the angle) — drawing the diameter through P, using the isosceles triangles formed by radii, and applying the exterior angle theorem.
- State and explain Thales' theorem as a corollary of the inscribed angle theorem.
- State and explain the inscribed quadrilateral theorem as a corollary of the inscribed angle theorem.
- Apply the paper-corner trick to find the diameter of a circle from a fragment that covers more than a semicircle.
- Identify the single underlying fact — all radii are equal — that powers all three results.

LAUNCH — THE BROKEN SHARD

Launch task (≈10–12 min)

"An archaeologist has found a fragment of a circular pottery vessel. She wants to estimate the circumference of the original — to get a sense of how large it was. Ultimately, we would like to be able to do this for any fragment, no matter how small. But we don't have all the tools for that yet. So today we're going to do what mathematicians often do: solve a simpler version of the problem first. Your shard still has more than half the original rim. See what you can figure out."

"We will not finish the full problem today — we will finish it in Lesson 8.6, when we have one more tool. But we will finish the simpler version. And we will discover the theorem that does the work — the inscribed angle theorem — which is the spine of the rest of this unit."

Give each group their shard. Frame honestly, then give them 2–3 minutes to work. Do not hint. Watch what happens.

What you will see:

- Tracing and rotating — students trace the arc, then rotate the fragment to complete the circle. Natural and clever. Acknowledge, then push: "You've drawn a full circle. Now how do you find its diameter exactly? An estimate won't do."
- Drawing chords — connect two points on the arc and try to find the center by perpendiculars. Right instinct — exactly what Lesson 8.6 will formalize.
- Trial circles — use a compass to find a circle that fits the arc. Acknowledge the instinct, but again: an estimate, not an exact value.
- Stuck — some groups will realize quickly they don't have a clean method. That is the right response — they are ready to learn one.

After 2–3 minutes, stop and say:

“Most of you have good instincts here. But to get the exact diameter — not an estimate — we need a theorem we haven't proved yet. We're going to spend the next part of class finding that theorem. I'll ask you to come back to the shard once you have it.”

TASK — WHAT IS THE RELATIONSHIP?

📋 Thinking Task — Discover the Inscribed Angle Theorem (≈25–30 min)

Move groups to vertical boards. Each group draws a large circle and marks the center. Give the task in two steps:

“Step 1: Pick any two points on your circle and label them A and B. Draw the angle AOB at the center — that is the central angle for arc AB. Measure it.”

“Step 2: Pick a third point P anywhere on the major arc. Draw the angle APB — that is an inscribed angle for the same arc. Measure it. What do you notice? Now move P to a different position on the major arc. Measure again. Can you make a conjecture?”

Once groups have a conjecture, push explicitly to the inductive→deductive move:

“You have just done inductive reasoning — measured several cases, noticed a pattern, made a conjecture. That is how mathematics often begins. But a conjecture is not a theorem. How would you convince a skeptic, using deductive reasoning, that this must always be true?”

What you will see (in order of sophistication):

Stage	What groups do	How to push
1. Measuring	Measure the two angles, get a ratio close to 2:1, assume coincidence	*“Try a different position for P. Try a different pair A, B. Does the ratio change?”*
2. Conjecturing	Realize the inscribed angle is always half the central angle. State as a conjecture.	*“You've made a conjecture using inductive reasoning. Now convince me it is always true. What would a deductive argument look like?”*
3. Explaining	Attempt extra lines; often draw the radius to P	*“What kind of triangle is OPA? Why? What does that tell you about its angles?”*
4. Proving	Use the isosceles triangle OPA and the exterior angle theorem to derive the 2:1 ratio	*“Write a clean proof. What happens if the center is outside angle APB? Does your argument still work?”*

The key nudge — draw the diameter through P

The key nudge is drawing the diameter through P. Once students draw the line from P through O to the far side of the circle, the problem usually breaks open: triangle OPA is isosceles (OP = OA = radius), so its base angles are equal. Calling those base angles α, the central angle AOP' (where P' is the far point) equals 2α by the exterior angle theorem. The same argument applies to the other side. Adding gives the full result. The three cases of the proof (center inside, on, or outside the angle) are important for completeness, but Case 1 is the one to get every group through.

Extension Ladder

Level 1: What happens when P is on the minor arc instead? Measure the inscribed angle. What is its relationship to the central angle now?

Level 2: What if AB is a diameter? What is the inscribed angle APB? Try several positions for P. State this as its own theorem.

Level 3: Connect four points on the circle to form a quadrilateral APBQ. Measure all four angles. What do you notice? Can you explain why?

Level 4 — Open: Come back to the broken shard. You now have a new theorem. Is there a clever way to use it to find the exact diameter of the original vessel — without finding the center, and without estimating?

Levels 2 and 3 are the corollaries — prefer student presentation

Level 2 (Thales' theorem) and Level 3 (inscribed quadrilateral) are the corollaries you will consolidate for the whole class. Groups that discover them independently before consolidation make for an ideal gallery walk and consolidation sequence — you can have them present rather than delivering the corollaries yourself. Level 4 is a preview of the paper-corner trick. If a group notices that a 90° inscribed angle always subtends a diameter, they have essentially invented Thales' theorem and can begin thinking about how to use it practically.

CONSOLIDATION — FROM CONJECTURE TO THEOREM

Begin with student presentations, not teacher notes. Select two or three groups in order of depth: first, a strong conjecture without proof; second, the beginning of a deductive argument; third, a complete proof if available. Only after student work do you write the formal version on the board.

The Theorem — for the Board

Inscribed Angle Theorem

An inscribed angle is half the central angle that subtends the same arc.

Equivalently: an inscribed angle equals half the intercepted arc (in degrees).

The Proof — Case 1 (center inside the inscribed angle)

Write the proof step by step. Ask students to supply each justification before you write it.

Given: Circle with center O. Points A, P, B on the circle. O lies inside ∠APB.

Prove: ∠APB = ½ ∠AOB.

Draw the diameter through P — extend PO to point D on the circle.

Triangle OPA: OP = OA (radii). So ∠OAP = ∠OPA = α. *[Why isosceles?]*

Exterior angle theorem: ∠AOD = ∠OAP + ∠OPA = 2α. *[Which theorem?]*

Triangle OPB: OP = OB (radii). So ∠OBP = ∠OPB = β. *[Why isosceles?]*

Exterior angle theorem: ∠BOD = ∠OBP + ∠OPB = 2β. *[Which theorem?]*

∠AOB = ∠AOD + ∠BOD = 2α + 2β = 2(α + β).

∠APB = α + β.

Therefore ∠APB = ½ ∠AOB. ■

After the proof, ask: “What was the single move that made this proof work?” Students should name it: drawing the diameter through P. Then ask: “Why could we always do that?

Corollary 1 — Thales' Theorem

Ask: "What if AB is a diameter? What does the inscribed angle theorem predict?" Students should get there immediately: central angle = 180°, so inscribed angle = 90°.

Thales' Theorem: If AB is a diameter of a circle, then for any point P on the circle (P ≠ A, B), the angle APB = 90°.

Proof: The central angle for a diameter is 180°. By the inscribed angle theorem, ∠APB = ½ × 180° = 90°. ■

Historical aside — Thales of Miletus

Thales of Miletus (c. 624–546 BC) is credited with being the first person in the Western tradition to prove geometric results from logical principles rather than observation alone. He predates Euclid by roughly 200 years. Whether Thales actually proved this particular theorem is disputed by historians, but his name is attached to it in every geometry course in the world.

This is the same Thales who measured the height of the Great Pyramid of Giza using similar triangles and shadow ratios. The very technique that we used in Unit 6. He measured his own shadow at the moment it equaled his height, then measured the pyramid's shadow at the same moment, and concluded the pyramid's height equaled its shadow length. The connection between Thales' shadow method and the inscribed angle theorem is not a coincidence — both reflect the same mathematical temperament: hidden measurements can be recovered from visible relationships, if you know the right theorem.

Corollary 2 — Inscribed Quadrilateral

If a group reached Extension Level 3, ask them to present first. Otherwise ask: "Some groups found something interesting about quadrilaterals inscribed in a circle. What did you notice?"

Inscribed Quadrilateral Theorem: Opposite angles of a quadrilateral inscribed in a circle are supplementary.

Proof: Angle A subtends arc BCD; angle C subtends arc DAB. Together these arcs make 360°. So 2∠A + 2∠C = 360°, giving ∠A + ∠C = 180°. ■

Ask: "Is this a new theorem, or is it the inscribed angle theorem wearing a different hat?" Answer: the latter. Students who see this structure will not need to memorize it separately.

THE PAPER CORNER CHALLENGE

Returning to the shard

Pose the challenge — or invite a group that got level 4 to present:

"You have a theorem that says an inscribed angle of 90° always subtends a diameter. You have a right-angle corner. You have your shard. Can you use these to find the exact diameter of the original vessel?"

Two minutes to try. The method: place the right-angle corner on the arc so both edges touch the rim. The two contact points are endpoints of a diameter by Thales. Draw the chord. Measure it. That is the exact diameter. Then: "What is the circumference? What was the original vessel's size?" If all shards came from the same circle, all answers should match.

Narrative Arc — Close of class — setup for Lesson 8.6

“This worked because your shard was large — you could fit a right-angle corner on it. What if the fragment were smaller than a semicircle? The paper-corner trick would fail. In Lesson 8.5 we will find a construction that works for any fragment, no matter how small. Today you have solved the problem for the easy case. That is still mathematics.”

This closing reframe is important. Students should feel the satisfaction of having solved something real, while also feeling the productive dissatisfaction of knowing the full problem is still open. Both feelings are authentic mathematical experiences.

✎ Students make notes on:

- Definitions: central angle, inscribed angle, intercepted arc — with a labeled diagram for each.
- The inscribed angle theorem stated precisely, plus the meaning of ‘subtend the same arc.’
- A sketch of the Case 1 proof in their own words — the key move (drawing the diameter through P) and why it worked.
- Thales' theorem, with an explanation of why it follows from the inscribed angle theorem — not stated as a separate fact to memorize.
- The inscribed quadrilateral theorem, with an explanation that opposite angles subtend arcs summing to 360°, so the inscribed angles sum to 180°.
- The paper-corner trick — in words — and which theorem justifies it.
- What all three results have in common: the single fact that all radii are equal, which produces the isosceles triangles in every proof.

Connection Points	**Draws from Lesson 8.1:** The definition of a circle (all points equidistant from the center) is what guarantees that drawing the diameter through P is always a valid move, and what makes the triangles OPA and OPB isosceles. The whole proof rests on it. **Draws from Unit 3:** The exterior angle theorem, isosceles triangle theorem, and triangle angle sum are all in active use here. The proof is a meeting point of Unit 3 (triangle theorems) and Unit 8 (circles). **Sets up Lesson 8.5:** The shard problem reappears with a smaller fragment, where the paper-corner trick fails. The chord-bisector theorem replaces it. **Sets up Lesson 8.6:** The chord-sine identity (and therefore the Law of Sines as a circle theorem) uses the inscribed angle theorem and Thales' theorem in one combined diagram. Students who own today's theorem will recognize the move tomorrow immediately.