Lesson 4.4 — Teacher Reference Notes

Where Fairness Lives: The Bisector Theorems

Unit 4: Triangle Theorems • Day 4 of 6

At a Glance	
	Pacing: Task 1: Beacon (~20 min) • Extension 1 (~15 min) • Task 2: Water Tower (~15 min) • Extension 2 (~10 min) • Gallery walk if time allows • Consolidation: definitions + both proofs (~20 min) • Notes + CYU begins (~10 min) **Materials:** Boards and markers **Nav instrument rows active:** Row 4 (Bisector Theorems) activates today — all three levels • Row 2 Advanced: perpendicular bisector as a derived result

OBJECTIVES

By the end of this lesson students can:

- Define perpendicular distance and equidistant precisely.
- State the angle bisector theorem and perpendicular bisector theorem as conditional statements, along with their converses.
- Prove both theorems using right triangle congruence.
- Use both theorems and their converses to find missing measurements and set up algebraic equations.
- Explain why the perpendicular bisector of a segment is a derived result from the altitude construction and the base angles theorem.

Narrative Arc — Open class — before Task 1

"In Unit 3 we used angle bisectors as a tool to prove things about triangles. In Lesson 4.2 we used the altitude construction. In Lesson 4.3 we proved the base angles theorem. Today all three of those ideas connect. We are going to ask: what does it actually mean to lie on a bisector? Not how to make one — what it guarantees."

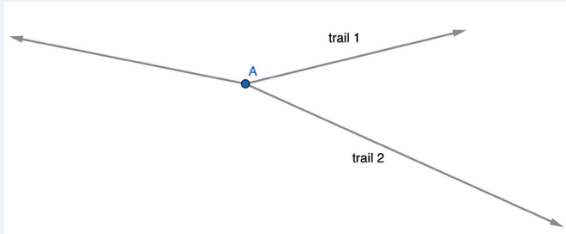
TASKS

Task 1 — The Search-and-Rescue Beacon (≈20 min)

Read aloud. Do not write on the board.

Prompt

Two ranger trails meet at a junction. The park service wants to place a single emergency beacon somewhere along the trails — not at the junction — so that it is equidistant from either trail. Draw the situation. Experiment with possible beacon locations. Where could the beacon be placed? Is there only one location, or are there many? What do all valid locations have in common? Record every conjecture — even ones you later cross out.



⚠️ **What to watch for**

The geometry: the two trails form an angle; the beacon must be equidistant (perpendicular distance) from both sides. Valid locations form a line — the angle bisector.

Groups testing individual points: ask 'does one good location suggest others? What do your successful points have in common?'

Groups that find the bisector line: push them to state the conjecture precisely as a conditional: 'if a point is on the angle bisector, then…'

Extension 1 — Prove It (given individually when groups have the conjecture)

Prompt

You've found that all fair beacon locations seem to lie on a specific line. State your conjecture precisely as a conditional statement. Can you prove it? What triangles can you see in your diagram? Does the converse hold — if a point is equidistant from both trails, must it lie on that line?

The proof uses AAS on the two right triangles formed by dropping perpendiculars from a point on the bisector to each side. The converse uses HL (or AAS) going the other direction.

Task 2 — The Water Tower (≈15 min, give to all groups once most have worked through Task 1)

Prompt

Two towns want to share a water tower. To keep the supply fair, the tower must be exactly the same distance from each town. Mark two towns on your board. Experiment with possible tower locations. Where could the tower go? Is there only one location, or many? What do all valid locations have in common? How is this situation similar to the beacon problem? How is it different?

The key distinction from Task 1

Task 1: distance is measured perpendicularly from a point to a line — equidistant from two lines.

Task 2: distance is measured directly from a point to another point — equidistant from two points.

Both produce the same kind of answer: a line of fair locations. The line in Task 2 is perpendicular to the segment connecting the two towns and passes through its midpoint — the perpendicular bisector.

Groups that find isolated points: ask ‘does one fair location suggest others nearby?’

Groups that find the perpendicular bisector: ask ‘how would you prove every point on this line is equidistant from both towns?’

Extension 2 (given individually when groups have the perpendicular bisector conjecture)

Prompt

You’ve found that all fair tower locations lie on a specific line. State your conjecture as a conditional statement. Prove it using triangle congruence. What triangles do you see? Does the converse hold?

The proof uses SAS on the two triangles formed by the midpoint and a point on the perpendicular bisector. The converse direction is also provable by SAS.

CONSOLIDATION — THREE PARTS

Part A — Establish the vocabulary before the theorems

Before formalizing either theorem, name the terms students were using informally during the tasks.

Perpendicular Distance	Equidistant
The perpendicular distance from a point to a line is the length of the segment from the point to the line that meets it at a right angle. It is the shortest distance from the point to the line.	A point is equidistant from two lines if its perpendicular distance to each line is equal. A point is equidistant from two points if its straight-line distance to each point is equal.

Part B — The Angle Bisector Theorem and Converse

State both theorem and converse, then work through the proof together.

Angle Bisector Theorem: If a point lies on the bisector of an angle, then it is equidistant (perpendicular distance) from the two sides of the angle.

Converse: If a point is equidistant from the two sides of an angle, then it lies on the angle bisector.

Angle Bisector Theorem — Proof

Given: Point P on the bisector of ∠ABC. PD ⊥ BA at D, PE ⊥ BC at E. Prove: PD = PE.

Statement	Justification
∠PDB = ∠PEB = 90°	*PD ⊥ BA and PE ⊥ BC — definition of perpendicular distance.*
∠DBP = ∠EBP	*P lies on angle bisector of ∠ABC — definition of angle bisector.*
BP = BP	*Reflexive property.*
△PDB ≅ △PEB	*AAS: two angles and non-included side (right angles, bisected angle, shared hypotenuse).*
PD = PE ■	*CPCTC — P is equidistant from both sides.*

Conditional form: If a point lies on the bisector of an angle, then its perpendicular distance to each side is equal.

Converse proof — HL or AAS

Given PD = PE (both perpendicular), prove BP bisects ∠ABC.

Using HL: right angle given, hypotenuse BP shared, leg PD = PE given. △PDB ≅ △PEB by HL. Therefore ∠DBP ≅ ∠EBP by CPCTC — BP bisects ∠ABC.

Part C — The Perpendicular Bisector Theorem and Converse

State both theorem and converse, then work through the proof together.

Perpendicular Bisector Theorem: If a point lies on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

Converse: If a point is equidistant from the endpoints of a segment, then it lies on the perpendicular bisector of that segment.

Perpendicular Bisector Theorem — Proof

Given: Segment AB with midpoint M. Line l passes through M perpendicular to AB. P is any point on l. Prove: PA = PB.

Statement	Justification
AM = MB	*M is the midpoint of AB — definition of midpoint.*
∠PMA = ∠PMB = 90°	*l ⊥ AB — definition of perpendicular bisector.*
PM = PM	*Reflexive property.*
△PMA ≅ △PMB	*SAS.*
PA = PB ■	*CPCTC — P is equidistant from both endpoints.*

Conditional form: If a point lies on the perpendicular bisector of a segment, then it is equidistant from the endpoints.

THE PERPENDICULAR BISECTOR AS A DERIVED CONSTRUCTION

Say this explicitly — it closes a thread from Lesson 4.2

Students have not constructed a perpendicular bisector as a standalone procedure — but they already have all the pieces. The altitude construction from Lesson 4.2 drops a perpendicular from an external point to a line. If that point is the apex of an isosceles triangle, the altitude lands on the midpoint of the base.

Why? The altitude proof in 4.2 showed △APH ≅ △AQH by SAS, giving PH = QH by CPCTC — so H is the midpoint of PQ. And the altitude is perpendicular to the base. Therefore: altitude from the apex of an isosceles triangle = perpendicular bisector of the base.

To construct the perpendicular bisector of any segment AB: treat AB as the base of an isosceles triangle by finding any two points equidistant from A and B (use the same compass opening from each endpoint). Connect them. The line through those two points is the perpendicular bisector. Students already know how to do every step of this — they just haven't named the result.

THE DEEPER IDEA — WORTH NAMING

What these theorems actually say

Both bisector theorems make the same fundamental claim: a bisector is not just a line that divides something in half — it is the precise set of all points that satisfy an equal-distance condition.

The angle bisector: the set of all points equidistant from two lines.

The perpendicular bisector: the set of all points equidistant from two points.

The constructions build these lines. The theorems tell you what membership in them means. This is the first time in the course that students encounter a geometric object defined as a set of points satisfying a condition rather than as a construction procedure. That shift — from 'here is how to make it' to 'here is what it guarantees' — is what turns a construction into a theorem.

DEFINITIONS REFERENCE — TEACHER COPY

Angle Bisector Theorem + Converse

Theorem: If a point lies on the bisector of an angle, then it is equidistant (perpendicular distance) from the two sides of the angle.

Converse: If a point is equidistant from the two sides of an angle, then it lies on the angle bisector.

Biconditional: A point lies on the bisector of an angle if and only if it is equidistant from the two sides.

Perpendicular Bisector Theorem + Converse

Theorem: If a point lies on the perpendicular bisector of a segment, then it is equidistant from the endpoints.

Converse: If a point is equidistant from the endpoints of a segment, then it lies on the perpendicular bisector.

Biconditional: A point lies on the perpendicular bisector of a segment if and only if it is equidistant from the endpoints.

STUDENTS MAKE NOTES ON

✎ **Students make notes on:**

- Perpendicular distance: definition with diagram
- Equidistant: two forms — from two lines (perpendicular distance) and from two points (straight-line distance)
- Angle bisector theorem: conditional + converse + biconditional + proof (AAS)
- Perpendicular bisector theorem: conditional + converse + biconditional + proof (SAS)
- The perpendicular bisector as derived from the altitude construction — why they already know how to do this

Connection Points	• Draws from Lesson 4.2: the altitude proof gives PH = QH by CPCTC — H is the midpoint, so the altitude from an isosceles apex is the perpendicular bisector of the base • Draws from Lesson 4.3: isosceles triangles are the geometric object that makes the perpendicular bisector derivation work • Draws from Unit 3: AAS and SAS are the shortcuts used in both proofs; CPCTC gives the equidistant conclusion • Draws from Unit 2: both theorems form biconditionals — the language and logical structure are directly from Unit 2 • Feeds into Unit 5: perpendicular bisectors and angle bisectors appear again in the study of quadrilateral diagonals