

Measuring the Inaccessible

Unit 6: Similarity • Day 4 (two class periods) • Day 1: Thales + outdoor • Day 2: Mirror task

At a Glance

Day 1 pacing: Thales task at boards (~25 min) • Outdoor investigation: shadow + one invented method (~40 min) • Return indoors: conversion + comparison (~20 min) • Error analysis discussion (~5 min)

Day 2 pacing: Mirror task Phase 1 empirical (~15 min) • Phase 2 diagram + conjecture (~20 min) • Phase 3 law of reflection proof (~15 min) • Phase 4 prediction before test (~15 min) • Consolidation: all three methods notes (~15 min) • CYU (~10 min or homework)

Materials Day 1: Boards and markers for Thales task • Protractors (optional, for angle-of-sight method)

Materials Day 2: One mirror per group (small, flat — ideally a tile or acrylic mirror, 10–15 cm) • Measuring tape

Source credit (Day 2 mirror task): Adapted from Dietiker, Kysh, Sallee & Hoey, Core Connections Geometry (CPM).

Nav instrument rows active: Row 3 (Indirect Measurement) activates across both days

OBJECTIVES

By the end of this lesson students can:

- Use the shadow method to find the height of an inaccessible object, identifying the two similar triangles and the AA~ justification.
- Use the mirror method to find the height of an inaccessible object, identifying the similar triangles and the law of reflection.
- Derive the law of reflection (angle of incidence = angle of reflection) as a consequence of similar triangles.
- Set up and solve indirect measurement problems using any of the three methods.
- Articulate the assumptions each method requires and identify possible sources of error.
- Convert informal body-based units to standard units using proportional reasoning.

The broader arc of this lesson — worth naming for yourself before Day 1

This lesson is the fourth in a series of geometry-as-measurement moments across the course. In Unit 2, students used Eratosthenes' method to measure the Earth's circumference. In Unit 3, they used congruent triangles to measure an uncrossable river (Napoleon's general method). In Unit 6 Lesson 6.1, Descartes' coordinate geometry allowed distance measurement on the plane and on the Earth. Today, students use similar triangles to measure something tall.

The throughline: geometry is not just about proving things. It is about measuring the world. Naming this arc explicitly in Day 2's consolidation is one of the strongest moments in the course.

DAY 1 — THALES AT THE PYRAMID

 **Narrative Arc** — Launch Day 1 — say this before the task

“Imagine you are standing in Egypt around 600 BCE. You are Thales of Miletus. The Pharaoh wants to know the height of the Great Pyramid. You may not climb it. You may not damage it. You have no modern tools — only your mind, your body, and what nature provides. How could you possibly measure it?”

Thales Task — Board Work (≈25 min)

Groups work at boards. Encourage bold proposals. Let impractical ideas surface — they reveal assumptions worth examining.

Your role as ideas emerge

When students ask for information: judge plausibility. If Thales could reasonably have known or measured it in 600 BCE, you may provide it. Ground is flat, objects stand upright, sunlight can be treated as parallel rays. Do not mention shadows unless students raise the idea first.

Thales' insight — guide toward this if no group reaches it after sustained effort

‘What if Thales didn’t try to measure the pyramid directly at all? What if he waited?’

He waited until his own shadow (or a vertical stick’s shadow) was exactly equal to his height. At that moment, the similar triangles have a 1:1 ratio. So the pyramid’s height equals the length of its shadow measured from the center of its base.

The geometry: both person and pyramid form right triangles with the ground. Sunlight rays are parallel — corresponding angles equal. $AA \sim$. At the special moment, one triangle has a 1:1 ratio, forcing the same ratio in the other.

Extension if time allows (Aristotle anecdote)

‘According to Aristotle, Thales once cornered the olive-press market to prove philosophers weren’t useless. Let’s pretend today is harvest day and Thales cannot wait for his shadow to match his height. Can you come up with another way he might have found the height?’

Valid alternatives: shadow at a known time of day (find the angle using a protractor-like tool), similar stick method at a different known ratio, line-of-sight method using a mirror or a pool of water. Accept any mathematically valid approach.

DAY 1 — OUTDOOR INVESTIGATION (≈40 MIN)

Transition with: ‘We may not be in ancient Egypt, but the same problem exists today. Let’s see if we can measure something we can’t reach.’

The outdoor constraint — state this before going out

No rulers, meter sticks, or tape measures outdoors. Students must use informal but repeatable units: steps taken toe-to-toe, forearm lengths, foot lengths, or similar.

Why: conversion itself is proportional reasoning. The same ratio tool that finds the missing height also converts units. Making this explicit reinforces that similarity is not a specialized technique but a general way of thinking about ratio.

⚠ During the outdoor investigation

Shadow method: most groups will use this immediately. When they do, ask: ‘What triangles are you creating? Which angles are equal? What assumptions are you making?’

Second method: students must invent one that does not use shadows. Resist rescuing them quickly — this productive struggle is where geometric reasoning happens, not applying a memorized procedure.

Line-of-sight / clinometer: many groups eventually invent a method where one student lies on the ground and aligns their view with the top of the object while another marks the angle. This is valid and worth celebrating.

Mirror method preview: if any group independently discovers placing a mirror on the ground and walking back until they see the top of the object, name it and save it for Day 2’s formal treatment.

DAY 1 — RETURN INDOORS: CONVERSION AND ERROR ANALYSIS (≈25 MIN)

Conversion as proportional reasoning

Students convert their body-based units to standard units. This is not a diversion — it is the same proportional reasoning used in similarity: 1 forearm length = x cm, so 14 forearm lengths = $14x$ cm. Name this explicitly.

Whole-class comparison after conversion. Ask:

- ‘How close are different groups’ answers? What explains the differences?’
- ‘Which assumptions from each method were hardest to satisfy? What error does each produce?’
- ‘The mathematics of similarity is exact. Real-world measurement is not. What does that mean for how confident we should be in our answer?’

Common sources of error to surface

Shadow method: ground not flat, shadows not measured from the same point in time, person not standing exactly vertical, shadow endpoint hard to locate precisely.

Line-of-sight: the eye is not at ground level (introduces a vertical offset), the alignment angle is hard to measure precisely without a protractor. Please see and use the wonderful

desmos demonstration created by the wonderful Matt Chadbourne:

<https://www.desmos.com/calculator/e2lafejcod>

Conversion: inconsistency in stride length or forearm length between measurements. Any ratio error compounds.

DAY 2 — THE MIRROR TASK (THE HYPNOTIST PROBLEM)

Source credit

Adapted from Dietiker, Kysh, Sallee & Hoey, Core Connections Geometry (CPM).

Emphasis adapted to foreground similarity, conjecture, and proof.

 **Narrative Arc** — Launch Day 2 — read this aloud

“Legend has it that if you stare into someone’s eyes in just the right way, you can hypnotize them into squawking like a chicken. The person being hypnotized needs to stand exactly 200 cm from a mirror on the floor and look straight at the mirror. Your job is to find where the hypnotist must stand so that when they stare at the mirror, they see the other person’s eyes.”

Phase 1 — Empirical Discovery (≈15 min)

Each group chooses one person to be the subject. That person stands exactly 200 cm from the mirror and looks into it. The rest of the group moves until they can look into the mirror and see the subject’s eyes.

No explanations required yet. The goal is simply to make it work. Once a group finds the position, they hold the configuration.

Phase 2 — Diagram, Measure, Explain (≈20 min)

Tell groups: ‘You found where to stand. Now explain why that position works.’

Groups return to boards and draw a diagram. Do not tell them what to include — let them decide what needs measuring. Eye height and distance to the mirror typically emerge, along with both people being perpendicular to the floor.

What to expect and how to respond

Most groups: will see two triangles, one on each side of the mirror, and conjecture they are similar.

Press for justification: do not confirm similarity. Ask: ‘Which angles are equal? Which sides correspond? How do you know?’

AA~ argument: both triangles have a right angle (person standing perpendicular to floor). The angle at the mirror is the key — it will be addressed in Phase 3.

Phase 3 — From Similarity to Reflection (≈15 min)

Once groups have argued similarity, ask:

‘If these triangles are similar, what must be true about the angles where the line of sight meets the mirror?’

The argument to develop

The line of sight from Person A to the mirror and from Person B to the mirror form two angles with the mirror surface. In the similar triangles, corresponding angles are equal. Therefore the angle of incidence (angle the incoming line of sight makes with the mirror surface) equals the angle of reflection (angle the outgoing line of sight makes with the mirror surface).

Only after groups have articulated and proved this should you name it: the law of reflection. In this lesson, it is not a rule to memorize. It is something students have derived from similar triangles. If you would like an explanation to avoid the empirical nature of how we got at this one, please check out the Desmos demonstration created by my super talented colleague Matt Chadbourne: <https://www.desmos.com/calculator/mmwnowmotr>

Angle of incidence and reflection — introduce the terms here

Angle of incidence: the angle between the incoming ray and the perpendicular to the mirror surface (the normal). Angle of reflection: the angle between the outgoing ray and the normal. The law: angle of incidence = angle of reflection.

Phase 4 — Prediction Before Testing (≈15 min)

Reset the situation. A different pair of people in the group. One person stands 200 cm from the mirror. Before experimenting, groups must calculate where the second person should stand using their diagram and the similar triangle proportion.

Only after committing to a prediction in writing may they test it with the mirror.

Why this phase matters

This transforms similarity from explanation into prediction. A student who can predict where to stand, write the proportion, solve it, and then verify with the mirror has genuinely understood what similar triangles are for.

CONSOLIDATION — ALL THREE METHODS (≈15 MIN)

Day 2 consolidation covers all three methods together. Use the Liljedahl note-making approach: groups construct notes at the boards, then copy to student note sheets.

 **Narrative Arc** — The bigger picture — name this in consolidation

“In Unit 2 we used parallel lines and angle relationships to measure the circumference of the Earth the way Eratosthenes did. In Unit 3 we used congruent triangles to measure an uncrossable river. Today we used similar triangles to measure things we cannot touch. Same tools, harder problems. Geometry is not just about proving things. It is about measuring the world.”

Method	Assumptions	Similarity shortcut	Proportion to set up
Shadow	Both objects perpendicular to ground. Sun's rays are parallel (treated as parallel, strictly true only if the sun is very far away).	AA~ — equal angles: right angle + same sun angle	person height / person shadow = object height / object shadow
Mirror	Both objects perpendicular to ground. Angle of incidence = angle of reflection (law of reflection — derived in Day 2).	AA~ — right angle + equal reflection angles	eye height / distance to mirror = object height / object distance from mirror
Line of sight / angle	Both objects perpendicular to ground. Line of sight is straight. The sight angle and the position form consistent similar triangles.	AA~ — right angle + equal sight angle	distance to known / known height = distance to unknown / unknown height

Worked example reference — shadow method

A student who is 1.5 m tall casts a shadow 2 m long. At the same time, a lamppost casts a shadow 10 m long. Find the height of the lamppost.

Triangles similar by AA~: right angles at base + equal sun angles.

Proportion: $1.5/2 = h/10 \rightarrow h = 7.5$ m.

Worked example reference — mirror method

A mirror is placed on the ground 2.5 m from a wall. A student stands 1.25 m from the mirror and sees the top of the wall. The student's eye height is 1.6 m. Find the height of the wall.

Triangles similar by AA~: right angles at base + equal reflection angles.

Proportion: $1.6/1.25 = h/2.5 \rightarrow h = 3.2$ m.

STUDENTS MAKE NOTES ON

Students make notes on:

- Shadow method: assumptions, diagram, AA~ justification, proportion to set up, worked example
- Mirror method: assumptions including law of reflection (derived, not assumed), diagram, AA~ justification, proportion to set up, worked example
- Angle / line-of-sight method: assumptions, diagram, AA~ justification, proportion to set up
- Sources of error for each method and how they affect the result
- Conversion: how informal units become standard units via proportion

Connection Points

- Draws from Lesson 6.3: AA~ is the shortcut used in every indirect measurement proof in this lesson — the lesson exists because of what was established in 6.3
- Draws from Unit 2: Eratosthenes' method (parallel rays, proportional reasoning) is the historical predecessor of today's shadow method
- Draws from Unit 3: Napoleon's general (congruent triangles to measure a river) is named explicitly in consolidation as part of the measurement arc
- Feeds into Lesson 6.5: converting body-based units reinforces proportional reasoning; the constant-ratio idea is the bridge to scaling laws
- The law of reflection derived here connects to physics and will reappear if students study optics