Unit 3: Proof and Congruence | CYU 3.4 — Differentiation Day

Navigation Instrument Rows 1–4

Check Your Understanding 3.4

Show all work. Use correct notation.

Basic	Intermediate	Advanced

Row 1 — Constructions and What They Prove → *Proof begins with what we can construct from axioms*

Spiral — Construction vs. Measurement

A student claims she has proven a line segment is the same length as another by measuring both with a ruler and getting the same number. A second student proves it by constructing a circle centered at one endpoint with radius equal to the first segment, and showing the second segment is also a radius of that circle.

(a) Which student has proven the segments are equal? Explain.

(b) What postulate or common notion justifies the second student's argument?

(c) Explain why the first student's method is a drawing argument and not a proof.

Row 2 — Postulates vs. Propositions → *The foundation: what we accept vs. what we must earn*

Spiral — SAS as Postulate vs. Proposition

In Euclid's Elements, SAS is Proposition 4. In this course, we accept it as a postulate.

(a) What is the difference between treating SAS as a proposition vs. treating it as a postulate?

(b) What are we giving up by accepting it as a postulate?

Row 3 — Triangle Congruence Shortcuts → *The geometric facts we draw from to build arguments*

Basic

1. Identify the congruence shortcut or write "Not enough information."

(a) $\triangle ABC$ and $\triangle DEF$: $AB=DE$, $BC=EF$, $AC=DF$.

(b) $\triangle PQR$ and $\triangle XYZ$: $\angle P=\angle X$, $PQ=XY$, $\angle Q=\angle Y$.

(c) $\triangle GHI$ and $\triangle JKL$: $GH=JK$, $\angle H=\angle K$, $HI=KL$.

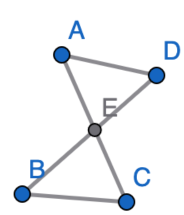
(d) $\triangle MNO$ and $\triangle PQR$: $MN=PQ$, $NO=QR$, $\angle O=\angle R$.

(e) $\triangle STU$ and $\triangle VWX$: $\angle S=\angle V$, $\angle T=\angle W$, $\angle U=\angle X$.

Intermediate

3. For each diagram, state which shortcut proves the triangles congruent. Identify each congruent part and state how you know it (given, vertical angles, shared side, etc.).

(a)



Advanced

5. In two triangles: $\angle A=\angle D$, $AB=DE$, and $\angle C=\angle F$.

(a) Is this sufficient to prove congruence? If so, which shortcut? If not, why not?

(b) Does it matter which angles are $\angle A$ and $\angle C$ relative to side AB? Explain carefully.

6. A student proposes: "SSA — if two sides and the angle opposite the longer side are congruent, the triangles must be congruent."

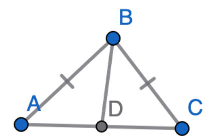
2. Write the one additional piece of information needed for the named shortcut.

(a) Given ∠A=∠D and ∠B=∠E. What else is needed for AAS?

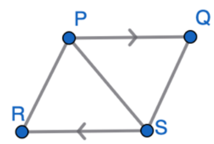
(b) Given AB=DE and BC=EF. What else is needed for SAS?

(c) Given ∠P=∠X and PQ=XY. What else is needed for ASA?

(b) △ABC with D the midpoint of AC. BD drawn. Given: AB=BC.



(c) Given: PQ∥RS and PQ=RS.



4. A student writes: "△ABC ≅ △DEF by SSS because all three sides look equal."

(a) What is wrong with this argument?

(b) What would a valid SSS argument require?

(c) Rewrite the argument correctly, stating what would need to be given.

(a) Is this student correct? Investigate using a diagram or argument.

(b) If the claim is true, explain why the 'longer side' condition is important.

(c) How does this relate to what we know about SSA?

Row 4 — Reading a Diagram → *Knowing what is given vs. what must be earned*

Basic

7. For each claim, write "Free from diagram" or "Must be given."

(a) ∠AEB = ∠CED when AC and BD intersect at E.

(b) AB ∥ CD in quadrilateral ABCD.

(c) AE=EC when E appears to be the midpoint of AC.

(d) ∠ABD + ∠DBC = 180° when A, B, C are collinear.

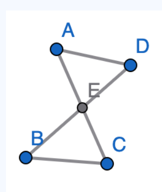
(e) AB = AB, where AB is a shared side of two triangles.

(f) ∠PQR = ∠STU in two triangles that look similar.

8. A student lists these "givens":

Intermediate

9. Segments AC and BD intersect at E. No marks given.

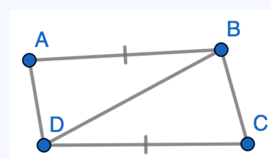


List all information you can state without a given. Identify the source of each piece.

Then: if you were additionally told AE=CE, which shortcut could prove two of the triangles congruent? Which triangles, and what is the complete list of congruent parts you would cite?

Advanced

12. Quadrilateral ABCD with diagonal BD drawn. Given: AB=CD.



Flawed proof: "Since AB=CD (given) and BD is shared, and since the angles at B and D look equal, the triangles are congruent by SAS."

(a) Identify every flaw in this proof.

(b) What additional given information would make SAS valid?

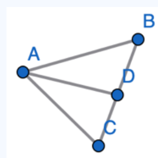
- The triangles share a common side.
- Two angles look equal because the triangles look symmetric.
- One side appears bisected by the shared side.

For each: (a) is it valid to use in a proof? (b) if not, what would make it valid?

10. A proof contains: "∠ABD = ∠CBD because BD bisects ∠ABC."

(a) What must appear in the given for this step to be valid?
(b) A student writes this step without it being given. What error have they made?
(c) Rewrite the setup so the step is valid.

11. △ABC with cevian AD drawn to point D on BC. No marks given.



Write every congruence or relationship you can state with certainty. Name your justification for each.

A student claims AD bisects ∠BAC. Can they use this without it being given? Explain.

(c) Is there a different shortcut that could work with AB=CD and BD shared, plus some additional given? If so, which?

13. A student correctly identifies three pairs of congruent parts, then writes △ABD ≅ △CBE.

(a) What does the order of letters in a congruence statement tell us?
(b) If the correct correspondence is A↔C, B↔B, D↔E, rewrite the statement correctly.
(c) Why does the order matter? What error would a reader make using the original statement?

Extension — Preview of Lesson 3.5

For students who have worked through the Advanced problems on Rows 3 and 4. This is a preview of the proof progression work beginning in Lesson 3.5. No formal proof structure has been taught yet — approach this as a reasoning exercise.

Given: ABCD is a parallelogram (opposite sides are parallel and equal). Diagonal AC divides it into △ABC and △CDA.

(a) List every piece of information you can state about the two triangles — from the given and from the diagram. Label the source of each.
(b) Do you think the two triangles are congruent? Which shortcut might apply?
(c) What is stopping you from writing a complete proof right now? What would you need to know how to do?

Answers — CYU 3.4

Row 1 Spiral — Construction vs. Measurement

(a) The second student has proven the segments are equal. Both segments are radii of the same circle — equal by definition of circle and Common Notion 1. The first student has only shown the segments measured the same on one occasion.

(b) Postulate 3 (a circle may be drawn with any center and any radius) justifies drawing the circle. Common Notion 1 justifies concluding the segments are equal.

(c) Measuring with a ruler relies on a physical instrument; it does not follow from the postulates. It shows segments appear to be equal; it cannot prove they must be equal within the axiomatic system.

Row 2 Spiral — SAS as Postulate vs. Proposition

(a) Proposition: must be proven from more basic axioms before use. Postulate: accepted as a starting point without proof — a rule of the system.

(b) We give up the complete logical chain from Euclid's axioms to this result, and the experience of seeing why SAS must be true.

Row 3 — Triangle Congruence Shortcuts

1a. SSS

1b. ASA

1c. SAS

1d. Not enough information (SSA)

1e. Not enough information (AAA)

2a. A non-included side: AB = DE (or BC = EF or AC = DF)

2b. The included angle: ∠B = ∠E

2c. The included side or a second angle: ∠Q = ∠Y (for ASA, the included side PQ = XY is given; need second angle)

3a. △ABE ≅ △CDE by SAS: AE=CE (not given — need given) or by ASA: ∠AEB=∠CED (vertical, free) plus shared angle plus one more given. Accept: identifies vertical angles as free; notes what additional given is needed.

3b. △ABD ≅ △CBD by SSS: AB=CB (given), BD=BD (shared/reflexive), AD=CD (D is midpoint of AC, given). Or by SAS: AB=CB (given), ∠ABD=∠CBD if given — but midpoint of AC doesn't give angle bisection. Accept SSS with all three parts justified.

3c. △PQS ≅ △RSQ by SAS: PQ=RS (given), ∠PQS=∠RSQ (alternate interior angles, PQ∥RS), QS=SQ (shared). Accept.

4a-c. (a) Appearance alone cannot justify a claim in a proof. (b) All three pairs of sides must be explicitly given or proven. (c) Accept any correct rewrite naming what would need to be in the given.

5a. Yes — AAS. Two angle pairs (∠A=∠D, ∠C=∠F) force the third, and the non-included side AB=DE is given.

5b. Yes — the position of the angles relative to the side determines whether the information is AAS or something else. If ∠C is not adjacent to AB in a way that makes the third angle forced, the correspondence could differ.

6a-c. (a) Correct — the ASS (or hypotenuse-leg for right triangles) condition does guarantee congruence when the angle is opposite the longer of the two given sides. (b) The longer side forces a unique position for the second side — it cannot swing to a second location. (c) Standard SSA fails because the second side can swing; ASS avoids this by requiring the angle to oppose the longer side, eliminating the ambiguous case.

Row 4 — Reading a Diagram

7a. Free from diagram (vertical angles)
7b. Must be given
7c. Must be given
7d. Free from diagram (linear pair)
7e. Free from diagram (reflexive property)
7f. Must be given
8a. Shared side: valid (reflexive/free from diagram).
8b. Equal angles by symmetry: not valid. Would need the angles explicitly given or provable from prior steps.
8c. Bisected side: not valid. Would need an explicit given that the shared segment bisects the other.
9. Free information: ∠AEB=∠CED (vertical), ∠AED=∠BEC (vertical), ∠AEB+∠BEC=180° (linear pair), ∠BEC+∠CED=180° (linear pair). All justified by diagram structure. Accept any well-reasoned response.
10a. The given must state: BD bisects ∠ABC.
10b. The student assumed from appearance (the line looks like it bisects the angle) without a geometric justification or given.
10c. Given: BD bisects ∠ABC. Then by definition of angle bisector: ∠ABD = ∠CBD.
11. Free: AD=AD (reflexive). No other congruences are free without given. Student cannot use 'AD bisects ∠BAC' without a given.
12a. Flaw: 'angles at B and D look equal' is not valid — must be given or derived from free information. SAS requires two sides and the included angle, all established.
12b. Need: ∠ABD = ∠CDB (or some angle information from a given).
12c. If additionally given BD bisects AC (or midpoint info), SSS could work if all three sides can be established.
13a. The order of letters encodes the correspondence: first vertex of one triangle corresponds to first vertex of the other, and so on.
13b. △ABD ≅ △CBE should be △ABD ≅ △CBE — with A↔C, B↔B, D↔E. Correct statement: △ABD ≅ △CBE.
13c. A reader using the wrong statement would apply CPCTC with wrong correspondences, concluding e.g. AB≅CB and BD≅BE instead of the correct pairs.