Lesson 8.3 — Teacher Reference Notes

Arc Length and Sector Area

Unit 8: Circles • Day 3 of 7

At a Glance	
	Pacing: Thinking task — the fraction idea + open extension (~30–35 min) • Gallery walk (~5 min) • Making meaningful notes (~25–30 min) • CYU (~10–15 min) **Materials:** Vertical whiteboards • Rulers and compasses • Calculators available but not required for the thinking task **Nav instrument row active:** Row 3 (Arc length and sector area → Applying proportional reasoning) activates today. Spirals to Rows 1 and 2 in the homework. **Key setup note:** Draw a circle on the board with a clearly marked central angle (90°, 120°, or 60° work well, use a clean fraction of 360°). Shade the sector. Do not write any formulas on the board before the task. The whole point is that students derive the formulas from the fraction idea themselves.

OBJECTIVES

By the end of this lesson students can:

- Explain that a central angle of $\theta°$ cuts off a fraction $\theta/360$ of the circle — for both arc and sector.
- Derive (not memorize) the arc length and sector area formulas as fraction × whole: $L = (\theta/360)\cdot 2\pi r$ and $A = (\theta/360)\cdot \pi r^2$.
- Explain why these formulas work for any circle, by connecting to Lesson 8.2 (all circles are similar).
- Use the formulas in both directions: given r and θ find L and A; given L or A find a missing angle or radius.
- (Advanced) Recognize that $A/L = r/2$ for any sector, and use it to find r from L and A together.

TASK

📋 Thinking Task — What Fraction Is This? (≈30–35 min)

Draw a circle on the board with a clearly marked 90° central angle. Shade the sector. Then give the task verbally:

"I have shaded a slice of this circle. The central angle is 90°. Without any formulas, decide: what fraction of the circle is the shaded region? What fraction of the outer edge (the circumference) belongs to the shaded arc? How long is that arc? How large is that sector area? The radius is 6 cm. Show all your reasoning on the board."

Do not give formulas. Do not mention arc length or sector area by name yet. Most groups will identify 90° as 1/4 of 360° and conclude: arc = 3π cm, sector = 9π cm². The job is to push them to generalize.

Extension Ladder — For Groups That Move Quickly

Give verbally as groups are ready, one at a time.

Level 1: What if the central angle is 120°? Find the arc length and sector area. What if it is 45°? Make a general rule.

Level 2: A circle has radius 10. A sector has area 25π. What is the central angle? Explain how you found it.

Level 3: An arc has length 8π. The central angle is 144°. What is the radius? What is the area of the sector?

Level 4: Connect the two radii from level 3, what is the area of the sector (a segment) that remains.

Level 5 — Open: A sector has arc length L and sector area A. Without being told the radius or the angle separately: can you find the radius using only L and A? Can you find the angle?

The hidden structure in Level 5 — $A/L = r/2$

Level 5 is genuinely challenging. The key insight: $L = (\theta/360)\cdot 2\pi r$ and $A = (\theta/360)\cdot \pi r^2$ share the factor $(\theta/360)$. Dividing gives $A/L = r/2$, so $r = 2A/L$. Then θ can be found from either formula. Students who discover this have found a beautiful structural relationship — the ratio of area to arc length gives the radius. This is well beyond the standard expectation and should be celebrated. If a group reaches it, they should present at consolidation.

CONSOLIDATION

Select one group that worked purely from the fraction idea (without formulas) and one that wrote the general formula. Ask both to present. The fraction-only group makes the structure visible; the formula group shows the compact expression. Both are useful.

✎ Students make notes on:

- What an arc is. What a sector is. A labeled diagram of both.
- Why a central angle of θ° corresponds to fraction θ/360 of the circle.
- The arc length formula $L = (\theta/360)\cdot 2\pi r$ and the sector area formula $A = (\theta/360)\cdot \pi r^2$, with the structural explanation — fraction × whole — in their own words.
- Why these formulas work for any circle, not just this one. Connect to Lesson 8.2.
- A worked example forward: $r = 9$, $\theta = 80°$ → find L and A.
- A worked example reverse: $L = 5\pi$, $\theta = 120°$ → find r.

The structure — for the Board

Write this explicitly. It should be the organizing principle of the notes.

For a circle with radius r and central angle θ°:

Fraction of the circle = θ / 360

Arc length = (θ/360) · 2πr

Sector area = (θ/360) · πr²

The formula is not a rule to memorize. It is just: fraction × whole.

Three questions to ask the class after writing the structure:

- *"Why does this fraction work? What is the reason θ/360 gives the right fraction?"*
- *"Does this formula work for any circle? Why? What did we prove in Lesson 8.2 that guarantees this?"*
- *"If I gave you the arc length and asked for the radius, what would you do?"*

⚠️ **Common errors to watch for**

- Forgetting to divide by 360 — using the full circumference or area instead of the fraction.
- Mixing up arc length and sector area — students sometimes confuse which uses r and which uses r^2.
- Using diameter instead of radius in $2\pi r$.

Address these in consolidation directly. The fraction-times-whole framing prevents all three, because students who think ‘fraction of the whole circumference’ won't lose the $2\pi r$ structure.

Narrative Arc — End of class — transition to Lesson 8.4

“Today you derived two new formulas without memorizing anything, just by thinking about fractions and what we already proved about similarity. Tomorrow we change gears: an archaeologist has found a piece of a broken pottery vessel, and she wants to know how big the original was. We have some tools that will help — and we are going to discover one more theorem that opens it up completely.”

Connection Points	**Draws from Lesson 8.1:** The circumference $2\pi r$ and area πr^2 of the full circle are starting points. Students do not derive these today, but the 'whole circle' factor in each formula comes from them. **Draws from Lesson 8.2:** This is the cash-in for yesterday. The reason $\theta/360$ works for every circle is that all circles are similar to the unit circle, and the unit circle's arc and area scale linearly by r and r^2 respectively. The proportional reasoning students did yesterday is doing today's work. **Sets up Lesson 8.4:** The central angle is the entry point to the inscribed angle theorem tomorrow. Students will be comfortable thinking of angles as fractions of 360° — a habit of mind that makes inscribed angles immediately accessible. **Sets up Lesson 8.6:** The relationship between angle and arc — and especially between arc and chord — is foundational for the chord-sine identity. The fraction $\theta/360$ reappears constantly.