Lesson 3.1 — Teacher Reference Notes

Constructions as Proof

Unit 3: Proof and Congruence • Day 1 of 9

At a Glance	
	Pacing: Task 1 (~15 min) • Task 2 (~30 min) • Task 3 (~15 min) • Consolidation 4 moments (~25 min) • Notes + CYU begins (~5 min) **Materials:** Compass and straightedge for each student • Large demonstration compass (strongly recommended) • Boards and markers • Paper on tables for Tasks 2–3 (compass work is better on paper than boards) **Nav instrument rows active:** Row 1 (Constructions and What They Prove) and Row 2 (Postulates vs. Propositions) activate today **Key setup note:** Ask at least one group to leave their dragon definition on the board after Task 1. You will return to it during consolidation.

OBJECTIVES

By the end of this lesson students can:

- Explain why a precise definition does not guarantee an object exists.
- Construct an equilateral triangle using compass and straightedge and justify every step.
- Write Euclid's Proposition 1 as a conditional statement.
- Distinguish between a postulate and a proposition and explain why the distinction matters.
- Explain why the nullilateral triangle's construction failure is stronger than merely not finding a method.
- Explain why Proposition 2 must follow Proposition 1, and what this reveals about a deductive system.

Narrative Arc — Open class with this — before revealing any tasks

"In Unit 2 we built a logical system. We have axioms we have agreed to accept. We have conditional statements and the law of syllogism. We have a shared foundation. Now we start using it."

TASKS

Task 1 — Define a Dragon (≈15 min)

Say only: "In your groups, write a precise definition of a dragon."

Once most groups have a definition on the board, ask:
“Look at your definition. Could two groups follow it exactly and end up with completely different-looking dragons? And here is the deeper question: does your definition make the dragon real? Does defining something carefully mean it exists?”

Purpose — do not explain this yet

This task is about existence, not dragons. Students will write careful definitions and feel the gap: no amount of precision makes the thing real. The pivot you are building toward: in geometry, a precise definition does not guarantee the object exists within the axiomatic system. However, a construction can prove it does exist. Do not name this yet — let students feel the gap. The consolidation will close the loop.

Task 2 — Euclid’s Proposition 1 (≈30 min)

Write on the board:

Prompt

Your tools: a straightedge (no measurements, from postulate 1 and 2)). A compass (draws circles — any center, any radius from postulate 3).

Your task: Given a line segment, construct a triangle where all three sides are exactly the same length. Every step must be justified — what did you do, and which tool rule allows it?

Compass technique — share this before groups begin

Paper works better than whiteboards for compass work — use paper on tables. The best technique: hold the compass still and rotate the paper. This gives more control than swinging the compass arm over a large surface.

As groups work: direct them to the postulates they already know. Postulates 1 and 2 govern the straightedge; Postulate 3 governs the compass. When a group finds the construction, ask: “How do you know all three sides are equal? Can you prove it using only postulates and common notions?” Do not answer — hold for consolidation.

⚠️ Do not walk through Proposition 1 with the class yet

Let groups work through both Task 2 and Task 3 before any class-wide review. The full construction walkthrough belongs in consolidation after the nullilateral triangle task, so both results can be understood together.

Task 3 — The Nullilateral Triangle (≈15 min)

After groups have worked through Proposition 1, give them this:

Prompt

"I want to define a new object. A 'nullilateral triangle' is a triangle in which the sum of two sides is less than the length of the third side. Using your compass and straightedge, try to construct one."

Note: 'nullilateral' is not a standard mathematical term. It was coined specifically to sound authentic while pointing toward the idea that this type of triangle is an empty set — no triangles satisfy the definition. Students are unlikely to catch the reference.

What students will find — use the large demonstration compass for this

Students will try and fail. That failure is the lesson. Demonstrate for the whole class:

- When the two radii sum to more than the third side: the circles intersect — a triangle can be formed.
- When the radii sum exactly to the third side: the circles are tangent, meeting at one point on the third side. The 'triangle' degenerates to a straight line.
- When the radii sum to less than the third side: the circles never reach each other. No intersection. No triangle.

After the demonstration: "I gave you a precise definition. But you could not construct it. Does that remind you of anything we talked about at the start of class?"

CONSOLIDATION — FOUR MOMENTS

All four consolidation moments follow both tasks. Do not begin until groups have worked through both the equilateral triangle construction and the nullilateral triangle attempt.

Moment 1 — Does Construction Prove Existence? (Board Task First)

Before any class-wide discussion, give groups this question to write and discuss on their boards:

"We said: if we can construct something, it exists in our system. Is the converse true? If something exists in our system, can we always construct it? And: when we tried to construct a nullilateral triangle and failed — did that failure prove it cannot exist, or did it only show we haven't found a method?"

After a few minutes, bring the class together. Draw out both instincts. Then make the logic explicit:

The two cases to establish

The converse is false: A counterexample: one-third of any angle exists as a geometric object — we can reason about it and be certain it is real. But it is provably impossible to trisect an angle with compass and straightedge. Existence is real; construction is impossible.

The nullilateral triangle is the stronger case: The compass demonstration is not merely a failed attempt. It is a constructive impossibility argument: when the sum of two sides does not exceed the third, the two circles cannot intersect — not because we haven't tried hard enough, but because the geometry makes it impossible. No intersection means no third vertex. No third vertex means no triangle. This proves the object cannot exist in the system.

Moment 2 — Euclid's Proposition 1: Walk Through the Proof

Now walk through the construction and its proof with the class. Use the reference table below. Draw each step on the board as you go — do not just read the steps aloud. Model the flow chart proof format by drawing boxes and arrows as you build.

Euclid's Proposition 1 — Construction and Proof (Teacher Reference)

Step	What we do / claim	Justification
1	Draw a circle centered at A with radius AB.	*Postulate 3: a circle may be drawn with any center and any radius.*
2	Draw a circle centered at B with radius AB.	*Postulate 3.*
3	Let C be a point where the two circles intersect.	*The circles intersect because both have radius AB — they reach each other. This will be revisited when we study the Triangle Inequality.*
4	Draw segments AC and BC.	*Postulate 1: a straight line may be drawn between any two points.*
5	AC = AB (both radii of circle centered at A).	*Definition of circle: all radii of a circle are equal.*
6	BC = AB (both radii of circle centered at B).	*Definition of circle.*
7	Therefore AC = BC = AB.	*Common Notion 1: things equal to the same thing are equal to each other.*

Conclusion:

Proposition 1 (conditional form): If we are given a line segment, then we can construct an equilateral triangle on it.

Note for students: we also used the definition of a circle in Step 5–6. Euclid lists 23 definitions before his first postulate — definitions are not overhead, they are foundational to the argument. Point this out.

Moment 3 — Postulate vs. Proposition (Critical — Do Not Rush)

Ask: "When we drew that circle centered at A — did we prove we were allowed to do that? Or did we just use it?" Students will recognize we just used it. Name it: Postulate 3.

Then: "So what is Proposition 1? Are we accepting it without proof, or are we claiming it and then proving it?"

The two P-words — write both on the board side by side

Postulate: a statement accepted as true without proof. A rule of the system. We agree to play by it.

Proposition: a claim that is stated and then proven from postulates, common notions, and previous propositions. We earn it.

One framing that works: postulates are the rules the game comes with. Propositions are moves you prove are legal.

Both words start with P. They are not the same thing. Require students to write the comparison table in their notes before moving to Moment 4.

Moment 4 — Euclid's Proposition 2 (Cover in Consolidation)

Tell students: "Euclid's Proposition 2 proves we can transfer a length to a new location — something the postulates alone do not allow. But its proof uses an equilateral triangle. Why does that mean Proposition 1 must exist first?" Let students reason about it, then walk through the steps.

Euclid's Proposition 2 — Given: point A and segment BC elsewhere. Construct a segment from A equal to BC.

Step	What we do / claim	Justification
1	Draw segment AB.	*Postulate 1.*
2	Construct an equilateral triangle ABD on segment AB.	*Proposition 1 (just proven — this is why it must come first).*
3	Extend lines DA and DB beyond A and B.	*Postulate 2: a finite line may be extended.*
4	Draw a circle centered at B with radius BC. Mark where it meets line DB extended — call this point G.	*Postulate 3.*
5	Draw a circle centered at D with radius DG. Mark where it meets line DA extended — call this point L.	*Postulate 3.*
6	DL = DG (both radii of circle centered at D).	*Definition of circle.*
7	DB = DA (sides of equilateral triangle ABD).	*Proposition 1 / definition of equilateral.*
8	BG = AL (subtracting equal lengths DB and DA from equal lengths DG and DL).	*Common Notion 3: if equals are subtracted from equals, the remainders are equal.*
9	BG = BC (both radii of circle centered at B).	*Definition of circle.*

10	Therefore AL = BC.	*Common Notion 1: things equal to the same thing are equal to each other.*

Conclusion:

Proposition 2 (conditional form): If we have a point A and a segment BC located elsewhere, then we can construct a segment from A that is equal in length to BC.

Key point to name for students: once a proposition is proven, it becomes a tool you can cite in future arguments without re-proving it. Proposition 1 — proven today — is cited in the proof of Proposition 2. This is how a deductive system scales. Every proposition we prove this year is a tool available for everything that follows.

DEFINITIONS REFERENCE — TEACHER COPY

Drawing vs. Construction

Drawing: uses any tool, measurements allowed. Shows a possible idea. A convincing drawing can still be wrong. Proves nothing.

Construction: uses only compass and straightedge, following the postulates exactly. No measurements. Proves the object exists and is exactly what was claimed, within the axiomatic system.

Postulate

A statement accepted as true without proof. A rule of the axiomatic system. Examples: Postulate 1 (a straight line may be drawn between any two points), Postulate 3 (a circle may be drawn with any center and any radius), Common Notion 1 (things equal to the same thing are equal to each other).

Proposition

A claim that is stated and then proven from previously accepted postulates, common notions, and earlier propositions. Once proven, a proposition becomes a named tool available for all future arguments. Examples: Proposition 1 (equilateral triangle construction), Proposition 2 (length transfer).

STUDENTS MAKE NOTES ON

✎ Students make notes on:

- Drawing vs. construction: two-column comparison table (tools, what it shows, what it proves)
- Postulate vs. proposition: two-column comparison table (what it is, how we know it, one example each)
- Euclid's Proposition 1: conditional form, construction steps with justifications, proof that the triangle is equilateral
- Why Proposition 2 must follow Proposition 1, and what this shows about how a deductive system is built

Narrative Arc — End of consolidation — transition to Lesson 3.2

"Today we answered the question we opened with: no, a definition does not guarantee existence. Only a construction can prove existence within the system. And we built our first chain of conditional statements — each step justified, each conclusion earning the next. That structure is what proof looks like. The next lesson gives us more tools to build with."

Connection Points	• Draws from Unit 2: Euclid's axioms are the shared premises students accepted in Lesson 2.6. Postulates 1, 2, and 3 are used in the construction. • Draws from Unit 2: Common Notions 1 and 3 are used in both proofs. • Feeds into Lesson 3.2: the term 'postulate' is needed immediately when triangle congruence shortcuts are accepted as postulates. • Feeds into Unit 3+: every proof in this unit and beyond cites propositions proven before it. The habit of citing justifications starts here.