Row 1 — What Is a Circle?

P1-B	**Basic** → *What is a circle?*

A circle has equation $(x + 3)^2 + (y - 5)^2 = 49$.

a) State the center and radius.

b) Is the point (4, 5) inside, outside, or on the circle? Justify.

Answer:

a) Center $(-3, 5)$, radius 7.

b) Distance from $(-3, 5)$ to $(4, 5) = 7$ = radius. The point is on the circle.

P1-I	**Intermediate** → *What is a circle?*

Find the equation of the circle with center $(2, -1)$ that passes through the point (6, 2).

Then find the two x-intercepts of the circle (where it crosses the x-axis).

Answer:

Radius = $\sqrt{(6-2)^2 + (2+1)^2} = \sqrt{16+9} = 5$. Equation: $(x-2)^2 + (y+1)^2 = 25$.

x-intercepts: set $y = 0$. $(x-2)^2 + 1 = 25$. $(x-2)^2 = 24$. $x = 2 \pm 2\sqrt{6}$.

P1-A	**Advanced** → *What is a circle?*

Two circles have equations $x^2 + y^2 = 25$ and $(x - 7)^2 + y^2 = 16$.

Without finding the intersection points, determine whether the circles intersect, are tangent, or do not meet. Justify your answer using only the centers and radii.

Then explain what must be true about the distance between centers if two circles are externally tangent (touch at exactly one point, neither inside the other).

Answer:

Centers: (0,0) and (7,0). Distance = 7. Radii: 5 and 4. Sum = 9 > 7; difference = 1 < 7. Since the distance between centers is between $|r_1-r_2|$ and r_1+r_2, the circles intersect at two points.

External tangency: distance between centers = $r_1 + r_2$ (the circles just touch, each on the outside of the other).

Row 2 — All Circles Are Similar

P2-B	**Basic** → *All circles are similar*

A circle has radius 4. A similar circle has radius 10.

a) What is the similarity ratio?

b) If the smaller circle has circumference 8π, what is the circumference of the larger circle? (Do not use the formula — use the similarity ratio.)

Answer:

a) Ratio = 10/4 = 5/2.

b) Circumference scales by the ratio: $8\pi \times (5/2) = 20\pi$.

P2-A	**Advanced** → *All circles are similar*

A student claims: “All circles are similar, so all sectors of the same angle must also be similar, regardless of the radius.”

Is this claim correct? If so, explain what the similarity transformation is and what the similarity ratio is. If not, give a counterexample and explain what goes wrong.

Answer:

The claim is correct. Two sectors with the same central angle are similar: map one circle onto the other by a dilation (scaling factor r_2/r_1, center at the sector’s vertex). The sector angle is preserved because dilation preserves angles. The arc length and radius both scale by r_2/r_1. The similarity ratio is r_2/r_1.

Row 3 — Arc Length and Sector Area

P3-B	**Basic** → *Arc length and sector area*

A sector has radius 9 and central angle 80°.

Find the arc length and sector area. Leave answers in terms of π.

Answer:

Arc length = $(80/360) \cdot 2\pi(9) = (2/9) \cdot 18\pi = 4\pi$.

Sector area = (80/360) · π(81) = (2/9) · 81π = 18π.

P3-I	**Intermediate** → *Arc length and sector area*

A sector has arc length 5π and sector area 15π.

Find the radius and central angle of the sector.

Answer:

From Lesson 8.3: sector area = (r/2) × arc length. So 15π = (r/2) · 5π. r = 6.

Arc length = (θ/360) · 2πr. 5π = (θ/360) · 12π. θ/360 = 5/12. θ = 150°.

Row 4 — Central and Inscribed Angles

P4-B	**Basic** → *Central and inscribed angles*

Points A, B, C lie on a circle. The central angle AOB = 112°.

a) Find the inscribed angle ACB (C on the major arc).
b) A different point D also lies on the major arc. What is angle ADB?

Answer:

a) ∠ACB = 112°/2 = 56°.

b) ∠ADB = 56°. All inscribed angles subtending the same arc are equal.

P4-I	**Intermediate** → *Central and inscribed angles*

A cyclic quadrilateral ABCD (all vertices on a circle) has angle A = 3x + 10 and angle C = x + 30.

a) Find x and both angles.
b) If angle B = 2y + 5 and angle D = y + 25, find y and both angles. Verify that B + D = 180°.

Answer:

a) A + C = 180° (inscribed quadrilateral). 3x+10+x+30 = 180. 4x = 140. x = 35. A = 115°, C = 65°.

b) B + D = 180°. 2y+5+y+25 = 180. 3y = 150. y = 50. B = 105°, D = 75°. Sum = 180°. ✓

P4-A	**Advanced** → *Central and inscribed angles*

Prove that if two inscribed angles subtend the same arc, they must be equal.

Your proof should reference the inscribed angle theorem and explain why the arc — not the chord — is the key object.

Answer:

Let ∠APB and ∠AQB both subtend arc AB (with P and Q on the major arc). By the inscribed angle theorem, ∠APB = ½(arc AB) and ∠AQB = ½(arc AB). Since both equal half the same arc, they are equal. The arc is the key object because the inscribed angle theorem is a relationship between an angle and the arc it intercepts — not between an angle and a chord. Two different points on the arc intercept the same arc and therefore give the same angle.

Row 5 — Tangent Lines and Chord Theorems

P5-B	**Basic** → *Tangent lines and chord theorems*

A tangent from external point P touches a circle of radius 5 at point T. The distance PT = 12.

Find the distance from P to the center O.

Answer:

Right angle at T (radius-tangent). $OP^2 = OT^2 + PT^2 = 25 + 144 = 169$. OP = 13.

P5-I	**Intermediate** → *Tangent lines and chord theorems*

A chord AB of a circle with radius 10 is 16 cm long.

a) Find the distance from the center to the chord.

b) A second chord CD is 12 cm long and parallel to AB. Find the distance between the two chords. (Consider both cases: same side of the center, and opposite sides.)

Answer:

a) Half-chord = 8. Distance = $\sqrt{(100-64)} = \sqrt{36} = 6$.

b) Half-chord of CD = 6. Distance from center to CD = $\sqrt{100-36} = 8$. Same side: $|8-6| = 2$. Opposite sides: $8 + 6 = 14$.

Row 6 — Circles and Triangles: Closing the Loop

P6-B	**Basic** → *Circles and triangles: closing the loop*

A triangle has side $a = 14$ and the angle A opposite to it is 45°.

Find the circumradius R.

Answer:

$a = 2R \sin A$. $14 = 2R \sin 45° = 2R(\sqrt{2}/2) = R\sqrt{2}$. $R = 14/\sqrt{2} = 7\sqrt{2} \approx 9.90$.

P6-I	**Intermediate** → *Circles and triangles: closing the loop*

A triangle has sides $b = 6$, $c = 8$, and included angle $A = 35°$.

a) Find the area.
b) Find side a using the Law of Cosines.
c) Find the circumradius.

Answer:

a) Area = $\frac{1}{2}(6)(8) \sin 35° \approx 24 \times 0.5736 \approx 13.77$.

b) $a^2 = 36 + 64 - 96 \cos 35° \approx 100 - 78.64 = 21.36$. $a \approx 4.62$.

c) $R = a/(2 \sin A) \approx 4.62/(2 \times 0.5736) \approx 4.03$.

Synthesis Problems — Multiple Rows

These problems intentionally draw on more than one rubric row. Attempt them only after you are comfortable at Intermediate level across the rows they involve.

S1	**Synthesis** → *Rows 1 + 4*

A circle has equation $x^2 + y^2 = 100$. Three points on the circle are $A = (10, 0)$, $B = (-10, 0)$, and $C = (6, 8)$.

a) Verify that C lies on the circle.
b) AB is a diameter. What is angle ACB? Justify using a theorem, not just coordinates.
c) Find the arc length of the major arc AC, given that the central angle subtending AC is 2 × ∠ABC.
d) Find ∠ABC using coordinates (find the angle at B in triangle ABC).

Answer:

a) $6^2 + 8^2 = 36 + 64 = 100$. ✓

b) ∠ACB = 90° by Thales' theorem: AB is a diameter, C is on the circle.

c) First need ∠ABC. Vector BA = (20, 0), vector BC = (16, 8). cos(∠ABC) = (BA·BC)/(|BA||BC|) = (320+0)/(20 × √320) = 320/(20 × 8√5) = 2/√5. ∠ABC = arccos(2/√5) ≈ 26.57°. Central angle = 2 × 26.57° = 53.13°. Major arc AC subtends 360° − 53.13° = 306.87°. Arc length = (306.87/360) × 2π(10) ≈ 53.59.

d) See part c: ∠ABC ≈ 26.57°.

S2	**Synthesis** → *Rows 3 + 5 + 6*

An equilateral triangle is inscribed in a circle of radius R.

a) Find the side length of the triangle in terms of R.
b) Find the area of the triangle in terms of R.
c) Find the arc length of one arc between adjacent vertices.
d) A tangent is drawn at each vertex of the triangle. The three tangents form a larger equilateral triangle outside the circle. Find the side length of the outer triangle in terms of R.

Answer:

a) Each angle of an equilateral triangle = 60°. s = 2R sin 60° = R√3.

b) Area = ½ s^2 sin 60° = ½$(R\sqrt{3})^2$(√3/2) = ½ × $3R^2$ × √3/2 = $3R^2\sqrt{3}/4$.

c) Each arc subtends a central angle of 120°. Arc = (120/360) × 2πR = 2πR/3.

d) The tangent at each vertex is perpendicular to the radius at that vertex. By symmetry, the outer triangle is equilateral. The distance from the center to each side of the outer triangle = R (the tangent line is at distance R from the center). For an equilateral triangle with inradius r, the side length = 2r√3. Side = 2R√3.

S3	**Synthesis** → *Rows 4 + 6*

Triangle ABC is inscribed in a circle. Angle A = 50°, angle B = 70°, angle C = 60°. The circumradius is R = 9.

a) Find all three side lengths using the chord-sine identity.

b) Find the area of the triangle in two ways: using Area = ½bc sin A, and using R = abc/(4K). Verify they agree.

c) Two of the triangle's inscribed angles subtend the same arc. Identify which pair, and explain what this tells you about those angles.

Answer:

a) $a = 2(9) \sin 50° \approx 13.79$. $b = 2(9) \sin 70° \approx 16.91$. $c = 2(9) \sin 60° = 9\sqrt{3} \approx 15.59$.

b) Area = $\frac{1}{2}(16.91)(15.59) \sin 50° \approx \frac{1}{2}(263.6)(0.766) \approx 100.9$. Check: $R = abc/(4K)$. $9 = (13.79)(16.91)(15.59)/(4K)$. $4K = (13.79)(16.91)(15.59)/9 \approx 403.5$. $K \approx 100.9$. ✓

c) Any two angles of the triangle subtend different arcs (since all three angles are different here). However, if any two angles were equal, they would subtend the same arc. In this problem all three are distinct. A better observation: angle A subtends arc BC (the arc not containing A), and so on for each angle. No two angles subtend the same arc in a scalene triangle.