

The Birthday Cake Problem and the Quadrilateral Family

Unit 5: Quadrilaterals • Day 1 of unit

At a Glance

Pacing: Task 1: Birthday Cake (~30 min) • Task 2: Bucket Sort (~25 min) • Gallery walk (~8 min) • Consolidation Parts A–C (~20 min) • Notes (~7 min) • CYU is homework

Materials: Shape manipulative sets printed on card stock — one per group. Include convex quadrilateral, concave quadrilateral, parallelogram, rectangle, rhombus, square, trapezoid, isosceles trapezoid, kite. Boards and markers. Hierarchy diagram image ready to display after Part B.

Nav instrument rows active: Row 1 (Classification and Definitions) — all three levels activate today

Prepare before class: Read the full six-solution set below before teaching. Winkler's completeness argument is the most important moment in the task — know it cold. Here is a link to a visual:

<https://www.theguardian.com/science/2019/oct/21/did-you-solve-it-the-four-points-two-distances-problem>

OBJECTIVES

By the end of this lesson students can:

- Name, define, and sketch each member of the quadrilateral family.
- Place any quadrilateral correctly in the hierarchy and explain why it belongs there.
- Explain why properties of a broader category automatically transfer to subcategories — and give a counterexample to the reverse.
- Connect the hierarchy to the conditional logic structure from Unit 2.

 **Narrative Arc** — Open class — before Task 1

“Every unit so far has started with a question about what something is or how to prove it. This unit starts differently. Today we start with a constraint problem. You are going to encounter the shapes we care about as solutions to a puzzle — not as a list to memorize.”

TASK 1 — THE BIRTHDAY CAKE PROBLEM (≈30 MIN)

Prompt

You want to arrange four candles on a birthday cake. How many essentially different ways can you place the four candles so that there are no more than two distinct distances between any pair of candles? Draw every configuration you can find. Mark

the distances clearly. Two arrangements count as the same if one is a rotation or reflection of the other. When you think you have found them all, ask yourself: how do you know you haven't missed one?

Source

Peter Winkler, Mathematical Mind-Benders (A K Peters, 2007). Featured by Alex Bellos in The Guardian, October 21, 2019. Solution article with visual diagrams:
<https://www.theguardian.com/science/2019/oct/21/did-you-solve-it-the-four-points-two-distances-problem>

Winkler notes: 'Nearly everyone misses at least one solution, and for each possible solution, it's been missed by at least one person.'

⚠ Common errors to watch for during the task

Non-square rectangle: has three distinct distances (short side, long side, diagonal). Fails. Ask: how many distinct distances does a non-square rectangle have?

General rhombus: has three distinct distances (side, short diagonal, long diagonal). Fails. Only the specific $60^\circ/120^\circ$ rhombus works.

General isosceles trapezoid: typically has three or more distinct distances. Only the pentagon-derived trapezoid (Solution 6) works.

The completeness question — 'how do you know you haven't missed one?' — is the most important moment of the task. Do not skip it. Push groups to justify, not just list.

Solution	Shape	Distances	Key note
1 — Square	Square	$4A + 2B$ (sides A, diagonals B)	Only shape where all four sides equal and both diagonals equal. Any square works.
2 — Rhombus (specific)	Rhombus with $60^\circ/120^\circ$ angles	$5A + 1B$	Two equilateral triangles sharing an edge. Short diagonal = side length = A.
3 — Equilateral Δ + center	Triangle with center point	$3A + 3B$ (sides A, center-vertex B)	Centroid, circumcenter, and incenter coincide. The 4th point is at the center.
4 — Isosceles config.	Equilateral Δ + point on perp. bisector	$4A + 2B$ (different config from square)	4th point above one side on perp. bisector, at distance A from base vertices.
5 — Kite (specific)	Specific kite	$3A + 3B$ (different arrangement from sol. 3)	Not the same as solution 3 — different point arrangement gives the same distance count.
6 — Pentagon subset	4 vertices of a regular pentagon	$2A + 4B$ (sides A, diagonals B of pentagon)	Remove one vertex from a regular pentagon. Most commonly missed solution.

TASK 2 — THE BUCKET SORT (≈25 MIN)

Distribute shape manipulative sets. Groups stay at the boards.

Prompt

Here is a set of shapes. Sort them into buckets at the board. Draw each bucket as a large region. Label it with the property that defines membership. Draw each shape inside every bucket it belongs to. A shape can appear in more than one bucket. Every shape must go somewhere. Once you have your buckets, answer on the board: Does any bucket sit completely inside another bucket? If so, draw it that way. Which shape appears in the most buckets? Why? Is there a shape that fits in only one bucket? What is the difference between a convex and a concave quadrilateral?

Common productive buckets students create

- '4 congruent sides' — rhombus and square
- '4 right angles' — rectangle and square
- '2 pairs of parallel sides' — parallelogram, rectangle, rhombus, square
- 'at least 1 pair of parallel sides' — all parallelograms plus trapezoids
- 'diagonals bisect each other' — all parallelograms
- '2 pairs of consecutive equal sides' — kite and square

⚠ The buckets-within-buckets moment — this is the key move

When a group notices that every shape in their '4 right angles' bucket is also in their '2 pairs of parallel sides' bucket: ask them to rearrange the boards so the smaller bucket is drawn as a circle inside the larger one. Then: 'Can you write what you just drew as a conditional statement?'

The square will appear in more buckets than any other shape. That is exactly right. A square satisfies the most constraints, so it belongs to the most categories. Students who find this satisfying rather than confusing have understood something important.

The concave quadrilateral belongs to no special bucket except the most general one. Its presence forces students to confront what the word quadrilateral actually means at its broadest.

GALLERY WALK (≈8 MIN)

Direct student attention to specific boards. This is not a free browse. Key questions:

- 'This group has a bucket that sits inside another. Read the labels. What conditional statement does that picture represent? Is the converse true?'
- 'Which shape appeared in the most buckets? Is that always the square? Why?'
- 'Did any group put the concave quadrilateral in a bucket other than the most general one? Is that valid?'
- 'Two groups used different bucket labels but ended up with the same shapes inside. What does that tell you about those properties?'

Part A — Formal Definitions (students fill in note sheet as you go)

Work through each definition together. Students write the definition in their own words and draw a sketch in the corresponding box on the note sheet. Do not read the definitions aloud and ask students to copy — ask them to state the definition first from their bucket work, then refine it together.

The trapezoid definition requires an explicit choice:

Trapezoid definition — state your choice explicitly

Inclusive (recommended): at least one pair of parallel sides. Under this definition, every parallelogram is a special trapezoid. The hierarchy flows cleanly.

Exclusive (some older texts): exactly one pair. Under this definition, parallelograms are not trapezoids. This creates a less elegant hierarchy.

Be explicit with students about which definition you are using and why. Students who have used a different textbook may have seen the exclusive definition. Name the disagreement honestly.

Formal definitions reference for teacher use:

Term	Definition
Quadrilateral	A polygon with exactly four sides.
Convex quadrilateral	A quadrilateral in which all interior angles are less than 180° . Both diagonals lie inside the figure.
Concave quadrilateral	A quadrilateral with at least one interior angle greater than 180° (a reflex angle). One diagonal lies outside the figure.
Trapezoid	A quadrilateral with at least one pair of parallel sides. The parallel sides are called bases; the non-parallel sides are legs. (Inclusive definition: parallelograms are special trapezoids.)
Isosceles trapezoid	A trapezoid whose legs are congruent.
Parallelogram	A quadrilateral with two pairs of parallel sides.
Rectangle	A parallelogram with four right angles.
Rhombus	A parallelogram with four congruent sides.
Square	A parallelogram with four right angles and four congruent sides. Equivalently: a rectangle that is also a rhombus.
Kite	A quadrilateral with two pairs of consecutive congruent sides. The two pairs share vertices, not a common side between them.

Part B — The Hierarchy (build from bucket work, then display image)

Build the hierarchy at the board together, starting from what students found. Once the class agrees on the structure, display the prepared hierarchy diagram image.

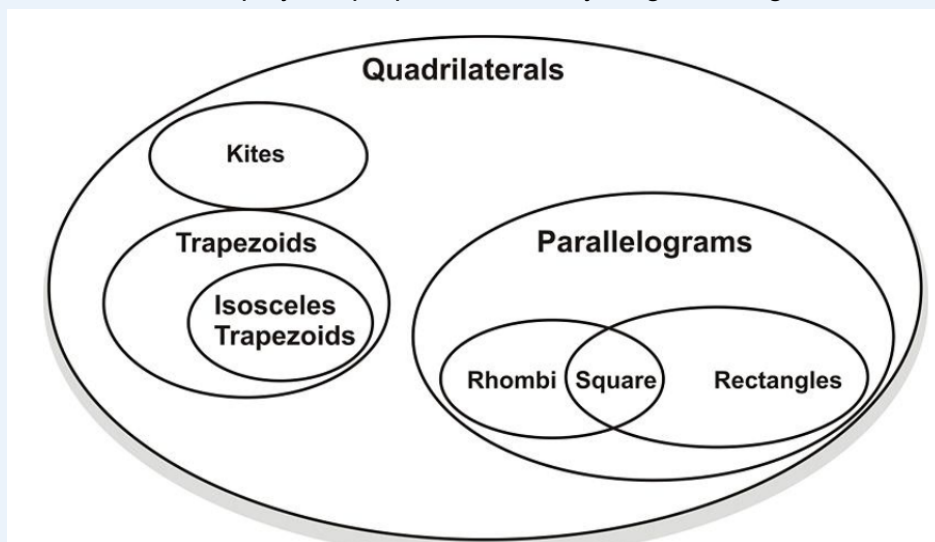


Image from: <https://www.ck12.org/section/classifying-quadrilaterals-%3a%3aof%3a%3a-parallel-lines-and-quadrilaterals-%3a%3aof%3a%3a-ck-12-foundation-and-leadership-public-schools-college-access-reader%3a-geometry/>

Part C — The Unit 2 Connection (name this explicitly)

Write on the board: If a quadrilateral is a rectangle, then it is a parallelogram. Ask: is the converse true?

Then say this:

'The conditional is true — every rectangle is a parallelogram because the definition of rectangle builds on the definition of parallelogram. The converse is false — a parallelogram with no right angles is a counterexample. This means: any property proven for all parallelograms is automatically true for rectangles, rhombi, and squares. No separate proof needed. But a property proven for rectangles does not transfer up to all parallelograms.'

Draw the Venn diagram: rectangle circle inside parallelogram circle. Properties of the outer circle are inherited by the inner circle. Properties of the inner circle do not reach the outer circle. This is the Unit 2 Venn diagram logic. Students were using this structure all lesson — name it now.

The Advanced column on Row 1 of the navigation instrument

A student who can state: 'any property of the outer category transfers automatically to subcategories because the subcategory is a subset of the outer category' — and who can give a counterexample to the reverse — has demonstrated the Advanced level on Row 1. Look for this during consolidation and name it explicitly when a student produces it.

STUDENTS MAKE NOTES ON

Students make notes on:

- All ten definitions: written in own words with sketches (note sheet)
- Hierarchy diagram: completed by student from class discussion, then confirmed with displayed image
- Two conditional statements from the hierarchy: one with true converse, one with false converse — with explanation or counterexample
- The Unit 2 connection: properties flow downward from broader to narrower categories, not upward

Narrative Arc — End of consolidation — transition to Lesson 5.2

“You just sorted shapes based on properties they share. The hierarchy you built is not a list of shapes to memorize — it is a logical structure. Every proof in this unit is going to use that structure: when we prove something about parallelograms, we get it for free in rectangles, rhombi, and squares. When we prove something about rectangles, it stays with rectangles. The direction matters.”

“Next lesson we ask: what can be proven about parallelograms? And the answer to that question will be the same answer every time in this unit — look for the hidden triangles.”

Connection Points

- Draws from Unit 0: the classification and bucket sort structure mirrors Lesson 0.1 — organizing by properties
- Draws from Unit 1: the shoe task (classification by properties) is the direct predecessor of the bucket sort
- Draws from Unit 2: the conditional logic of the hierarchy is the Venn diagram from Lesson 2.2b made geometric
- Feeds into Lesson 5.2: the parallelogram definition established here is the starting point for all four property proofs
- Feeds into every Unit 5 lesson: property inheritance means parallelogram proofs are automatically available for all special parallelograms