Lesson 8.6 — Teacher Reference Notes

The Chord-Sine Connection

Unit 8: Circles • Day 6 of 7

At a Glance	
	Pacing: Launch — open question from Unit 7 (~5–8 min) • Thinking task — triangle in a circle (~30–35 min) • Gallery walk (~5 min) • Consolidation — chord-sine, Law of Sines, (~20–25 min) • Notes (~10 min) • CYU (~10–12 min) **Materials:** Vertical whiteboards • Compasses, rulers, protractors • Optional: GeoGebra for dynamic exploration of the circumscribed circle **Nav instrument row active:** Row 6 (Circles and triangles: closing the loop → Synthesis with Units 6 and 7) activates today. Homework spirals across all six rows. **Key setup note:** Make sure students name the formula a/sin A = b/sin B = c/sin C before sending them to boards.

OBJECTIVES

By the end of this lesson students can:

- State and prove the chord-sine identity: for any chord of length ℓ subtending an inscribed angle θ, $\ell = 2R\cdot\sin\theta$.
- Explain that the Law of Sines is a theorem about the circumscribed circle — the common ratio a/sin A = b/sin B = c/sin C equals 2R, the diameter of that circle.
- Find the circumradius of a triangle given a side and the opposite angle, and find missing sides or area when given two sides and the included angle.
- (Advanced) Derive R = abc/(4K) by combining the circumradius and area formulas.

LAUNCH — THE OPEN QUESTION

Launch (≈5–8 min)

Recall the Law of Sines from Unit 7 with the class:

For any triangle ABC: a / sin A = b / sin B = c / sin C

Then pose the question:

“You proved this in Unit 7. You used it to find missing sides and angles. But there is a question we never answered. That common ratio — the number that a/sin A, b/sin B, and c/sin C all equal — what is it? Does it have a geometric meaning?”

Give students 30 seconds to think. Take a few guesses. Then say: “Today you are going to find out.”

TASK — A TRIANGLE IN A CIRCLE

📋 Thinking Task (≈30–35 min)

Move groups to vertical boards. Give the setup:

“Draw any triangle. Now circumscribe a circle around it — a circle that passes through all three vertices. You know how to find the circumcenter from Lesson 8.5

“Label the triangle ABC, the circumradius R, and the sides a, b, c opposite to angles A, B, C as usual.”

"Now: each side of the triangle is a chord of the circle. Each opposite angle is an inscribed angle. You have the inscribed angle theorem. What can you find?"

Do not say more. Let groups work.

The key nudge — I would give the students the circumscribed triangle with the two auxiliary constructions, with out any of the angles labeled.

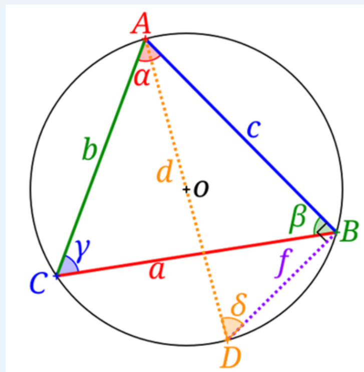


Image from Wikipedia

Extension Ladder

Level 1: Apply your result: a triangle has sides a = 8 and angle A = 30°. Find the circumradius R. Then find R for a triangle with a = 12 and A = 45°.

Level 2: Find the area of triangle ABC without using base × height directly. You know two sides and the included angle. Draw the altitude from B to AC and express it in terms of what you know.

Level 3: A triangle has sides a = 7, b = 9, and angle C = 60°. Find the area, the circumradius, and the side c (Law of Cosines).

Level 4 — Open: The circumradius formula gives R = a/(2 sin A). Can you find a formula for R in terms of all three sides a, b, c — with no angles? You may need to combine the circumradius formula with another formula from this lesson.

Level 4 is R = abc/(4K)

Extension Level 4 is genuinely hard. The answer is R = abc/(4K), where K is the area of the triangle. The route: use Area = ½ab·sin C and a/sin A = 2R to eliminate the angle. Groups that find this have connected circumradius, area, and side lengths in a single formula. That is a significant result. If a group reaches it, ask them to present at consolidation.

CONSOLIDATION — THE CHORD-SINE IDENTITY AND ITS CONSEQUENCES

Begin with a group that has the proof. Ask them to walk through it. Then ask the class: “What does this say about the Law of Sines? What was it all along?” Let students answer before you write anything.

The Chord-Sine Identity

For any chord of a circle with circumradius R, subtending an inscribed angle θ:

chord length = 2R · sin θ

The Law of Sines as a Circle Theorem

Apply the chord-sine identity to all three sides:

a = 2R · sin A ⟹ a / sin A = 2R

b = 2R · sin B ⟹ b / sin B = 2R

c = 2R · sin C ⟹ c / sin C = 2R

Therefore: a / sin A = b / sin B = c / sin C = 2R

The Law of Sines is a theorem about the circumscribed circle. The common ratio is the diameter of that circle.

The Proof — for the Board

Proof [edit]

As shown in the figure, let there be a circle with inscribed $\triangle ABC$ and another inscribed $\triangle ADB$ that passes through the circle's center $\mathbf{O}$. The $\angle AOD$ has a central angle of 180° and thus $\angle ABD = 90^\circ$, by Thales's theorem. Since $\triangle ABD$ is a right triangle,

$$\sin\delta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{c}{2R},$$

where $R = \frac{d}{2}$ is the radius of the circumscribing circle of the triangle.[3] Angles γ and δ lie on the same circle and subtend the same chord c; thus, by the inscribed angle theorem, $\gamma = \delta$. Therefore,

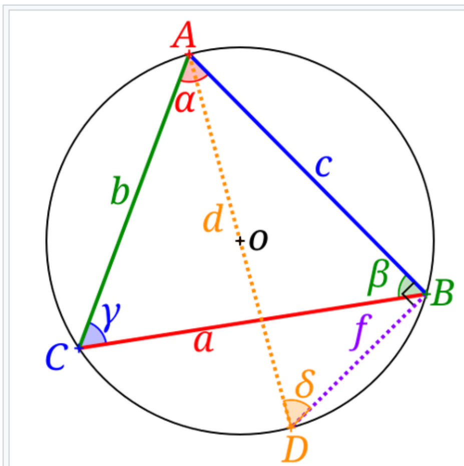
$$\sin\delta = \sin\gamma = \frac{c}{2R}.$$

Rearranging yields

$$2R = \frac{c}{\sin\gamma}.$$

Repeating the process of creating $\triangle ADB$ with other points gives

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.}$$



Deriving the ratio of the sine law equal to the circumscribing diameter. Note that triangle ADB passes through the center of the circumscribing circle with diameter d.

Image from Wikipedia

Ask: “Which step used the inscribed angle theorem? Which used Thales' theorem? Which used the definition of sine from Unit 7?” All three tools are present. The proof is a meeting point of the whole course.

> **Let this observation breathe — do not rush**
>
> Pause here. Ask: “In Unit 7, what did you think the Law of Sines was?” Students will say something like “a rule for solving triangles” or “a trig identity.” Then ask: “What is it now?” It is a statement about the relationship between a triangle and the circle that circumscribes it. Every triangle carries a circle with it, and the Law of Sines measures that circle.
>
> Do not rush past this moment. The feeling students should have is: ‘We have been using this theorem for weeks without knowing what it really was.’ That feeling, of retroactive understanding, of seeing a pattern you were using blindly now become transparent, is one of the most valuable experiences mathematics can offer. Name it. If time allows, ask: ‘What else from Unit 7 might be secretly a circle theorem?’

> **Narrative Arc — End of class — before review day**
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> *“You have closed a loop today that has been open since Unit 7. The Law of Sines is a circle theorem. Every triangle carries a circle with it, and that circle's diameter is what the Law of Sines measures. Take this with you. Tomorrow we review the unit — every theorem we have proved traces back to the same fact: all radii equal. The definition of a circle does the work.”*

> **Students make notes on:**
>
> - The master diagram: triangle ABC inscribed in its circumscribed circle, with sides a, b, c, angles A, B, C, circumradius R, the diameter used in the proof, and the right angle from Thales. This single diagram should contain everything.
> - The chord-sine identity: $\ell = 2R \cdot \sin\theta$, and the proof outline in their own words, naming the theorem used at each step (Thales, inscribed angle, definition of sine).
> - The Law of Sines stated again, this time with the explicit geometric meaning of the common ratio: 2R, the diameter of the circumscribed circle.
> - Area = $\frac{1}{2}ab \cdot \sin C$ with the altitude derivation — ‘where the sin comes from.’
> - A 'big picture' paragraph (3–5 sentences) connecting Units 6, 7, and 8: similarity, trigonometry, and circle geometry as a single connected web.

Connection Points	**Draws from Unit 6:** Similarity — the AA shortcut, the fact that ratios of sides are preserved under similarity, scaling laws. These are why the Law of Sines exists at all (the ratio a/sin A is the same for similar triangles). **Draws from Unit 7:** The Law of Sines (the formula itself), the Law of Cosines (used in some extensions), and the definition of sine in a right triangle. The Unit 7 work is necessary input — today's lesson interprets it, not derives it from scratch. **Draws from Lesson 8.4:** The inscribed angle theorem (Step 3 of the proof) and Thales' theorem (Step 2). Without yesterday's theorem, today's proof would be opaque. **Draws from Lesson 8.5:** The circumcircle is constructed using the perpendicular bisectors from yesterday — chord-bisector theorem in action. **Sets up Lesson 8.7:** The synthesis problems on review day all weave Units 6, 7, and 8 together. Today's master diagram is the visual scaffold students will use when they tackle a synthesis problem.