Lesson 7.7 — Teacher Reference Notes

The Law of Cosines

Unit 7: Trigonometry • Days 11–12 • Day 1: Deriving the Law • Day 2: Special Cases and the Ambiguous Case

At a Glance	
	Day 1 pacing: Opening framing (~3 min) • Thinking task: derive the law (~30 min) • Teacher-led derivation if needed (~15 min) • Notes at boards (~25 min) • CYU start **Day 2 pacing:** Pythagorean theorem as special case (~10 min) • Ambiguous case problem (~45 min) • Tool-selection discussion (~10 min) • Closing synthesis (~5 min) • CYU finish if needed **Materials:** Boards and markers • Calculators • Graph paper helpful for the ambiguous case (optional) **Nav instrument rows active:** Row 6 (Law of Cosines / tool selection) continues today **Day 1 key move:** Name the two segments of the base x and (whatever the side was already called −x), . This is the variable management insight. Most groups will drop the altitude correctly but stall on naming. Prompt: 'You have two right triangles. How many unknowns? How many equations?' **Year 1 hybrid structure:** 15 minutes of genuine discovery; teacher-led derivation if groups are stuck. The groups who don't fully derive still do real mathematical work. Math doesn't have a clock on it outside the classroom.

OBJECTIVES

By the end of this lesson students can:

- Derive the Law of Cosines from the Pythagorean theorem applied to a non-right triangle.
- Recognize the Pythagorean theorem as the special case where cos(90°) = 0 removes the correction term.
- Use the Law of Cosines to find an unknown side (SAS) or an unknown angle (SSS).
- Select the correct law (Sines or Cosines) based on what information is given.
- Identify the ambiguous case (SSA) and explain why it may yield two triangles, one, or none.

The central insight — name this before Day 1 begins

The Law of Cosines is not a new idea. It is the Pythagorean theorem applied to a non-right triangle. Students who know the Pythagorean theorem and who understand trig ratios have everything they need to derive it. The derivation is a synthesis, not new content.

Day 1 — Deriving the Law of Cosines

Narrative Arc — Opening framing — one sentence before releasing the task

“All year we have seen that creating hidden right triangles unlocks problems that seemed impossible. Keep that in mind today, even if you already have a visible triangle.”

DAY 1 — THINKING TASK (≈30 MIN)

Task prompt — written on board

The Pythagorean theorem says that in a right triangle, $c^2 = a^2 + b^2$. Would it be possible to find a similar relationship for any triangle — even one without a right angle? Try to find an equation that relates the three sides of a general triangle to each other.

Hits to give:

- **What happens when you create right triangles inside the general triangle?**
- **How many variables do you have?**
- **How many equations do you need?**

Guidance and common errors — in order of likelihood

The right move: all groups in Year 1 independently dropped an altitude. The direction is correct. The challenge is variable management.

The naming prompt: when groups have two right triangles but are stuck, ask: ‘You have two right triangles. What are the sides? How many unknowns do you have? How many equations can you write? Is that enough?’ The answer is: two unknowns (h and x), two equations. Substitution follows naturally.

Error 1 — $(a-x)^2 \neq a^2 - x^2$: the most common algebraic error in this derivation. When it appears, write ‘$(10-3)^2$ — what is that? Is it $10^2 - 3^2$?’ Students self-correct immediately once they see it numerically. Do not explain the error — let the example do it.

Error 2 — order of operations: when applying the law, students often compute $(b^2 + c^2 - 2bc)$ and then multiply by cos A, rather than computing $2bc \cdot \cos A$ first. Write: ‘$240 - 236 \cdot \cos(15°)$. How would you evaluate $240 - 236(3)$?’ Same structure.

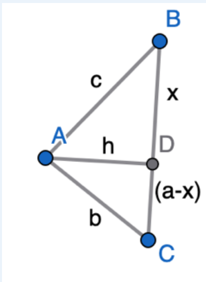
Partial derivation: groups who write $h = c \cdot \sin B$ as a step without finishing have done genuine work. Name this: ‘This step is essential. You have connected h to an angle. That’s exactly the move that makes the derivation work.’

Teacher-led derivation — run this if groups are stuck after 20 minutes

Begin: ‘Let me show you the key move, and then you complete the algebra.’ Name the foot H, the two segments x and (a−x). Write the two Pythagorean equations. Ask groups to subtract them and simplify. The step from h^2 back to the original sides is where you pause and ask: ‘What do you know about x?’

The Derivation — Step by Step (Teacher Reference)

Setup: Triangle ABC. Drop altitude h from A to side a. Label the foot H. Left segment BH = x, right segment HC = a − x.



Two Pythagorean equations from the two right triangles:
Top triangle (△ABD): $h^2 + x^2 = c^2$ …(1)
Bottom triangle (△ACD): $h^2 + (a-x)^2 = b^2$ …(2)

Subtract (1) from (2): $(a-x)^2 - x^2 = b^2 - c^2$
Expand: $a^2 - 2ax + x^2 - x^2 = b^2 - c^2 \rightarrow a^2 - 2ax = b^2 - c^2$
Rearrange: $b^2 = a^2 + c^2 - 2ax$

Recognize x using the cosine of angle B: $\cos B = x / c \rightarrow x = c\cdot\cos B$

Substitute: $b^2 = a^2 + c^2 - 2ac\cdot\cos B$
This is the Law of Cosines. ■

Special case: substitute $B = 90°$. $\cos(90°) = 0$, so the correction term $-2ac\cdot\cos B$ vanishes. $b^2 = a^2 + c^2$. The Pythagorean theorem is the Law of Cosines with the correction term switched off by the right angle.

DAY 1 — CONSOLIDATION: NOTES AT THE BOARDS (≈15 MIN)

Groups construct their own notes in response to these prompts. Post on the board:

Board prompts for notes

1. Write the Law of Cosines in all three forms (one for each side as the subject). State what information you need to use each form.
2. Show how the Pythagorean theorem is the special case when C = 90°.
3. Give one example where you would use the Law of Sines and one where you would use the Law of Cosines.

✎ Students make notes on:

- The Law of Cosines: all three forms (a^2 =, b^2 =, c^2 =) with a labeled diagram

- The Pythagorean theorem as special case: why cos(90°) = 0 removes the correction term
- The rearranged form for finding angles: $\cos A = (b^2 + c^2 - a^2) / 2bc$

Day 2 — Special Cases and the Ambiguous Case

DAY 2 — OPENING: PYTHAGOREAN THEOREM AS SPECIAL CASE (≈10 MIN)

Post this problem: a triangle with sides 3, 4, and included angle 90°. Apply the Law of Cosines. Show that the result is exactly the Pythagorean theorem.

Say this after working through it

‘We discovered the Pythagorean theorem millennia before the Law of Cosines was written down. We found the special case first, then found the general rule that contained it. This is the not an abnormal pattern in mathematics and science. Newton’s Law of Universal Gravitation is the special case of General Relativity. The ideal gas law is the special case of the Van der Waals equation. The simple version is not wrong; it is the general version with extra conditions switched off.’

DAY 2 — THE AMBIGUOUS CASE PROBLEM (≈35 MIN)

Task prompt — post on board

In a triangle: A = 30°, a = 6, and b = 10 (b is adjacent to A). Find the triangle. If there is more than one valid triangle, find all of them. At some point, ask yourself: am I assuming anything? If so, is that assumption justified?

⚠ Teacher notes for the ambiguous case

The 30-60-90 temptation: many groups will see A = 30° and immediately try 30-60-90 ratios. Prompt: ‘Are you assuming anything?’ Let them pursue the assumption. It will eventually fail when they check the Law of Sines on their results. Failing productively is the lesson.

The correct reading: this is SSA (two sides and a non-included angle). SSA does not guarantee a unique triangle — which is exactly why it is not a congruence shortcut (Unit 3). There are two valid triangles here.

The Unit 2 connection: when a group says ‘well, assuming…’ — name it explicitly. They are introducing a new premise without justification. In exploration this is fine. In a proof, every assumption must be flagged and tested. This is what Unit 2 was building toward.

The Unit 3 connection: SSA is not a congruence shortcut. This problem shows geometrically why: two non-congruent triangles can share the same SSA data.

DAY 2 — TOOL SELECTION DISCUSSION (≈10 MIN)

After the ambiguous case, have students make a tool selection guide at their boards. If you don't have time post the tool-selection table and work through it together. Ask students to identify the given information before deciding which law to use.

Configuration	Use	Why
Two angles + one side opposite a given angle	**Law of Sines**	An angle-side pair exists (find the third angle first if needed).
Two sides + non-included angle	**Law of Sines (careful: ambiguous case)**	An angle-side pair exists. May yield two triangles, one, or none.
Two sides + included angle	**Law of Cosines**	No angle-side pair. The included angle sits between the known sides.
Three sides	**Law of Cosines (rearranged for angle)**	No angle known. Rearrange: $\cos A = (b^2 + c^2 - a^2) / 2bc$.

DAY 2 — CLOSING SYNTHESIS (≈5 MIN)

Say this — if time permits

'This one problem — the ambiguous case — used the Law of Sines, the Law of Cosines, the triangle angle sum theorem, and the linear pair theorem. Something from almost every unit this year. We built knowledge on knowledge, and it carried us to a place we could not have reached in September. The ambiguous case is not a complication. It is a proof that the tools we built are powerful enough to reveal genuine subtlety.'

Connection Points	• Draws from Unit 1: the Pythagorean theorem (Lesson 1.2) is the foundation of the derivation; students who proved it in Unit 3 are now deriving the general form • Draws from Unit 3: SSA is not a congruence shortcut — the ambiguous case is the geometric reason why, now made explicit • Draws from Unit 2: the 'well, assuming…' moment connects directly to the logic of conditional statements and the requirement to justify every premise • Draws from Lesson 7.6: the Law of Sines is needed to solve the ambiguous case triangles once B is found; both laws work together in Day 2 • Feeds into Lesson 7.8 (decision tree): the tool-selection table built today becomes the core of the decision tree students will formalize in the final lesson