

Triangles Worth Remembering

Unit 7: Trigonometry • Days 6–7 • Day 1: Deriving Exact Values • Day 2: Practice with Exact Values

At a Glance

Day 1 pacing: Part 1: 45-45-90 derivation (~20 min) • Part 2: 30-60-90 derivation (~25 min) • Consolidation: notes at boards (~20 min) • Begin CYU if time allows

Day 2 pacing: Groups reproduce the side ratios from scratch (~10 min) • Complete any unfinished Day 1 notes (~10 min) • Practice problem bank from boards (~40 min) • CYU (~20 min or homework)

Materials: Boards and markers • No calculators

Nav instrument rows active: Row 4 (Special Right Triangles) activates today — all three levels

No calculator Day 1: State this explicitly at the start. The point is derivation, not computation. Any student who reaches for a calculator is solving the wrong problem.

Year 1 algebra warning: Students repeatedly wrote $(b-x)^2 = b^2 - x^2$. When this appears, write a numerical example on the board: $(10-3)^2$. Ask them to expand. Once they see it numerically, most self-correct.

OBJECTIVES

By the end of this lesson students can:

- Derive the exact side ratios for 45-45-90 triangles using the Pythagorean theorem and the isosceles structure.
- Derive the exact side ratios for 30-60-90 triangles using the equilateral triangle construction.
- Use exact values to find side lengths in special right triangles without a calculator.
- Rationalize denominators where needed ($\sqrt{2}/2$ form rather than $1/\sqrt{2}$).
- Use the special triangle values as estimation anchors to check reasonableness of other trig calculations.
- Prove the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ using a right triangle with hypotenuse 1.

The epistemic point — name this before Day 1 begins

A student who has proved these values is in a different position from one who has memorized them. If they forget, they can reconstruct. And they understand that mathematics sometimes delivers perfect answers — not approximations, not decimals, but exact geometric necessities. That is what today is about.

Board prompt — both parts given simultaneously

Your calculator gives decimal approximations. Today we want exact values. No decimals anywhere in your proof

Part 1: A right triangle has one acute angle of 45° . The hypotenuse is 1. Find the exact length of both legs. No decimals. What do you know about this triangle's shape? What does that tell you about the legs? Write a proof showing the exact values. Every step must be justified.

Part 2: A right triangle has a 30° angle. The hypotenuse is 1. Find the exact length of both legs. Your proof must include a diagram and justify every step.

45-45-90 — teacher guidance

The isosceles structure: If both acute angles are 45° , what kind of triangle must this be? Students should arrive at isosceles — the legs are equal. Pythagorean theorem: $x^2 + x^2 = 1^2 \rightarrow 2x^2 = 1 \rightarrow x = 1/\sqrt{2} = \sqrt{2}/2$.

If groups stall: 'If both acute angles are 45° , what kind of triangle is this?'

Rationalizing: Many groups will stop at $1/\sqrt{2}$. Introduce rationalizing as a convention — multiply top and bottom by $\sqrt{2}$ to get $\sqrt{2}/2$. Hold a mini-session for multiple groups simultaneously; this is more efficient than individual coaching.

Connect to 7.1: Group 4 was assigned 45° in Lesson 7.1. Their class table entry is this value. Name the connection explicitly.

30-60-90 — teacher guidance

The hint: Do not give the equilateral triangle hint too quickly. The struggle is the lesson. After 5 minutes of genuine effort: 'Think about equilateral triangles.'

After the hint: Press hard on the key justification step: 'Why does the altitude split the base exactly in half?' Push for the full argument: the equilateral triangle is isosceles with respect to any pair of sides, so the altitude from any vertex bisects the opposite base (by SAS or the base angles theorem from Lesson 4.3).

Scaling shortcut: Students who scale immediately to hypotenuse = 2 (rather than 1) avoid fractions entirely. Honour this approach — it shows mathematical fluency. The final ratios are the same.

The algebra error: If they write $(\text{something})^2 = b^2 - x^2$ expanding incorrectly, write $(10-3)^2$ on the board and ask them to evaluate. Self-correction follows.

DAY 1 — CONSOLIDATION: NOTES AT THE BOARDS (≈20 MIN)

Groups construct their own notes in response to these prompts. Post on the board:

Board prompts for consolidation notes

1. Draw and label both special right triangles (hypotenuse = 1). Mark all three side lengths exactly.
2. Write the exact values of sin, cos, and tan for 30°, 45°, and 60°.
3. Write a complete proof for the 45-45-90 triangle. Every step justified.
4. Write a complete proof for the 30-60-90 triangle starting from the equilateral triangle.
5. In 2–3 sentences: how do these triangles serve as estimation anchors for other trig values?

Notes — Special Right Triangles (teachers version)

The 45-45-90 Triangle

Why it works:

A right triangle with one 45° angle must have both acute angles equal to 45° (since $180^\circ - 90^\circ - 45^\circ = 45^\circ$). This makes it isosceles: both legs are equal. Let each leg = 1, hypotenuse = x

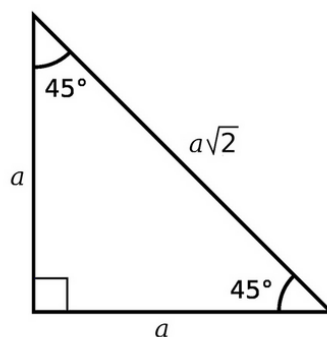
Pythagorean Theorem:

$$1^2 + 1^2 = x^2$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

Given: A right triangle with a 45° angle in it (an isosceles right triangle) and legs equal to x , then the hypotenuse = $x\sqrt{2}$



Example: Given a right triangle with a 45° angle in it and a hypotenuse of 1, find the length of the two legs.

$$x^2 + x^2 = 1^2$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

	sin	cos	tan
45°	$\sqrt{2}/2 \approx 0.707$	$\sqrt{2}/2 \approx 0.707$	1

The 30-60-90 Triangle

Why it works:

Start with an equilateral triangle of side length 1 (all angles 60°). Draw an altitude from one vertex to the opposite side. Because the triangle is equilateral (and therefore isosceles), this altitude is also the perpendicular bisector of the base, splitting it into two equal halves of length $1/2$. It also bisects the apex angle from 60° into two 30° angles. Each resulting right triangle has angles 30-60-90.

Pythagorean Theorem:

$$(1/2)^2 + h^2 = 1^2$$

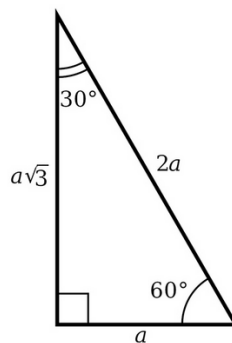
$$1/4 + h^2 = 1$$

$$h^2 = 3/4$$

$$h = \sqrt{3}/2$$

If we multiply everything by 2, to eliminate the fractions we get legs 1 and $\sqrt{3}$, hypotenuse 2

Given: a 30-60-90 right triangle if the short leg is a , then the longer leg is $a\sqrt{3}$ and the hypotenuse is $2a$



Example: Given a right triangle with a 30° angle in it and a hypotenuse of 1, find the length of the two legs.

Shorter leg:

$$a^2 + (a\sqrt{3})^2 = 1^2$$

$$a^2 + 3a^2 = 1$$

$$4a^2 = 1$$

$$a^2 = \frac{1}{4}$$

$$a = \frac{1}{2}$$

Longer leg: $\frac{1}{2}\sqrt{3} = \frac{\sqrt{3}}{2}$

	sin	cos	tan
30°	$1/2 = 0.5$	$\sqrt{3}/2 \approx 0.866$	$\sqrt{3}/3 \approx 0.577$
60°	$\sqrt{3}/2 \approx 0.866$	$1/2 = 0.5$	$\sqrt{3} \approx 1.732$

Notice: $\sin(30^\circ) = \cos(60^\circ)$ and $\sin(60^\circ) = \cos(30^\circ)$. This is the identity $\sin(\theta) = \cos(90^\circ - \theta)$ — the origin of the word 'cosine' (complement's sine). When two angles are complementary, opposite and adjacent simply swap roles.

PRACTICE WITH EXACT VALUES

Practice problem bank — post one at a time at the boards. No calculators.

Problem	Exact answer
45-45-90 with hypotenuse 10. Find both legs exactly.	$5\sqrt{2}$ each
45-45-90 with each leg = 7. Find hypotenuse.	$7\sqrt{2}$
30-60-90 with hypotenuse 8. Find both legs.	Short = 4, long = $4\sqrt{3}$
30-60-90 with short leg 6. Find hypotenuse and long leg.	Hyp = 12, long = $6\sqrt{3}$
30-60-90 with long leg $\sqrt{3}$. Find short leg and hypotenuse.	Short = 1, hyp = 2
45-45-90 with leg = $\sqrt{2}$. Find hypotenuse.	2
Equilateral triangle, side 10. Find exact height.	$5\sqrt{3}$
Square with diagonal 6. Find exact side length.	$3\sqrt{2}$
<i>All answers are exact. If a student reaches for a calculator: 'You have everything you need from the triangles you proved.'</i>	

STUDENTS MAKE NOTES ON

Students make notes on:

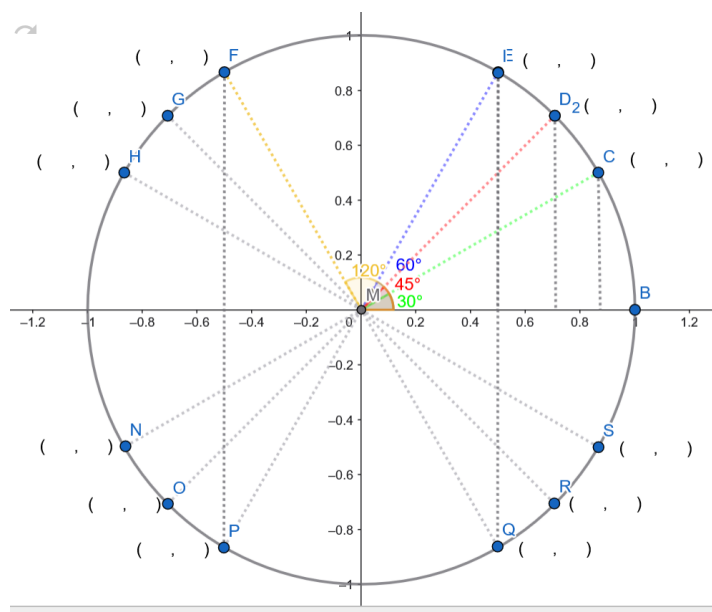
- 45-45-90 triangle: labeled diagram + full Pythagorean proof (legs = $\sqrt{2}/2$ when hyp = 1)
- 30-60-90 triangle: labeled diagram + full equilateral triangle proof (sides $1/2$, $\sqrt{3}/2$, 1 when hyp = 1)
- Exact values table: sin, cos, tan for 30° , 45° , 60° — students write from memory after the proof, not copied from a reference
- Estimation anchor principle: since sin is increasing from 0° to 90° , any value between two known angles is between their sines
- Rationalizing the denominator: why $1/\sqrt{2} = \sqrt{2}/2$ and how to perform the step

Open Day 2 with this

Launch

"Here is a circle of radius 1 centered at the origin. Angles are marked around the circle. Your task is to find the exact coordinates of every labeled point on the circle. You may not use a calculator. Use what you proved yesterday, and think carefully about the symmetry of the circle."

Do not add further instructions. The first quadrant (30° , 45° , 60°) should be immediately accessible from Day 1 work. The challenge is using symmetry to reason about the remaining quadrants.



What to Expect and Teacher Moves

Most groups will complete the first quadrant quickly. The key intellectual move is recognizing that the coordinates in other quadrants are reflections of the first-quadrant values, with sign changes determined by which quadrant the point is in.

- **If a group is stuck on the second quadrant:** "What is the x-coordinate of a point at 150° ? Draw the right triangle this point makes with the x-axis. What angle does it make? How does the triangle compare to one you already know?"
- **If a group assigns signs incorrectly:** "Is the point at 120° to the left or right of the y-axis? What sign should its x-coordinate have?"
- **If a group asks about 90° , 180° , 270° , 360° :** "These are the axis points. The circle has radius 1. Where does the circle cross each axis?" No right triangle is needed — just geometric reasoning.
- **For groups that finish early:** See the Pythagorean Identity extension below.

First Quadrant Reference (for teacher)

Angle	x-coordinate (= $\cos \theta$)	y-coordinate (= $\sin \theta$)
0°	1	0
30°	$\sqrt{3}/2$	$1/2$
45°	$\sqrt{2}/2$	$\sqrt{2}/2$
60°	$1/2$	$\sqrt{3}/2$
90°	0	1

The remaining three quadrants repeat the same absolute values with sign changes determined by the quadrant (x negative in Q2 and Q3; y negative in Q3 and Q4). A full answer key is provided at the end of this lesson.

Board Extension — The Pythagorean Identity

For groups that complete the unit circle task before the end of class.

"Verify that $\sin^2(30^\circ) + \cos^2(30^\circ) = 1$. Is this a coincidence? Check it for 45° and 60° as well. Then try to prove it must be true for any acute angle θ . What does this have to do with a right triangle with hypotenuse 1?"

What to expect: Groups that have been careful with their unit circle work will see it quickly — the point $(\cos \theta, \sin \theta)$ lies on a circle of radius 1, so by the distance formula, $\cos^2 \theta + \sin^2 \theta = 1^2 = 1$. For students who haven't framed it that way, ask them to draw the right triangle inside the unit circle with hypotenuse 1 and legs $\cos \theta$ and $\sin \theta$, then apply the Pythagorean theorem.

Why it matters: The Pythagorean Identity is one of the most used identities in all of mathematics — it reappears in precalculus, calculus, physics, and engineering. Students who derive it here from geometry will encounter it in precalculus not as a formula to memorize but as a fact they already proved.

Follow with CYU.

 **Narrative Arc** — End of lesson — transition to Lesson 7.5

"You now have a complete trig toolkit: ratios for any angle from the class table, the ability to solve in both directions, and exact values for three special angles. Lesson 7.5 brings everything together — you'll practice choosing the right tool for each problem. The goal is not speed. It is clarity: read the structure of the problem, select the appropriate tool, and execute without needing to think about which step comes next."

Connection Points

- Draws from Lesson 7.1: the 45° group (Group 4) has already measured these values — today they prove them exactly
- Draws from Lesson 4.3: the base angles theorem justifies why the equilateral triangle's altitude bisects the base (the equilateral triangle is isosceles)
- Draws from Unit 1: the Pythagorean theorem is the algebraic engine of both proofs
- Feeds into Lesson 7.5: the special right triangle values reduce calculator dependence in the mixed practice session
- CYU Q8 introduces the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ — a foundational precalculus result that follows naturally from a right triangle with hypotenuse 1